## *Chapter 7: Relativistic Treatment*

In the previous chapters, we have been using a mixture of relativistic and non-relativistic formalism. For instance, we have treated particles non-relativistically while describing the classical scalar field relativistically. This complication has arisen because the focus of our attention is the non-relativistic Bohm model, whereas the classical field examples we have drawn from textbooks as guiding illustrations all obey Lorentz covariant equations. Furthermore, discussions of energy-momentum tensors in books are nearly always formulated relativistically. For example, the only expression used for $T^{\mu\nu}{}_{particle}$ is $\rho_0 m u^\mu u^\nu$. Such presentations have a certain elegance whereas, as seen from equations [6-24a] to [6-24d] earlier, developing a non-relativistic treatment for $T^{\mu\nu}$ is messy and more tedious because of the need to keep track of separate expressions for $T^{ij}$, $T^{i0}$, $T^{0i}$ and $T^{00}$.

The situation becomes more critical in attempting to construct a particular formulation of Noether's theorem that will satisfy our present needs. A non-relativistic approach is more difficult and becomes unclear, whereas a relativistic one is found to be comparatively straightforward. For this reason, we will adopt a policy in the present chapter of presenting a fully relativistic treatment. The non-relativistic results that we will eventually need can then be obtained at the end by taking the non-relativistic limit.

To pursue this plan, it will be necessary to make temporary use of a relativistic version of Bohm's model before taking the limit. A suitable model for this purpose has, in fact, been formulated by Louis de Broglie[1]. While his model has certain contentious features compared with Bohm's original model, these features will not have any bearing on the present discussion because they do not affect the validity of our treatment and they vanish in the non-relativistic limit.

In the present chapter, it will be shown that a version of de Broglie's model incorporating energy and momentum conservation can be constructed in a straightforward way.

### 7.1 De Broglie's Model

In keeping with Bohm's approach, de Broglie presumes that a particle has a definite trajectory at all times. However, while Bohm's model is based on the Schrodinger equation, de Broglie's formulation involves the Klein-Gordon equation instead:

$$\frac{\hbar^2}{2m}[\partial_\mu \partial^\mu \phi + (\frac{mc}{\hbar})^2 \phi] = 0 \qquad [7\text{-}1]$$

where $\phi$ is the Klein-Gordon wavefunction. (The dimensional factor $\frac{\hbar^2}{2m}$ has been included here for ease of comparison with equations [7-8] and [7-17] later.) From chapter 3 (and using the notation $\psi = Re^{iS/\hbar}$), the basic postulate of Bohm's model is equation [3-8] for the particle's momentum **p**:

$$\mathbf{p} = \nabla S \qquad [7\text{-}2]$$

Using the analogous notation $\phi = Re^{iS/\hbar}$ in the Klein-Gordon case, the basic equation of de Broglie's relativistic model is:

$$p^\mu = -\partial^\mu S \qquad [7\text{-}3]$$

where now $p^\mu$ is the particle's 4-momentum. Equations [7-2] and [7-3] are sufficient for the minimalist versions (see chapter 3) of Bohm's and de Broglie's models, respectively. If one wishes to go further and introduce a "quantum potential" Q into each model (as is convenient for our purposes), the appropriate expressions are as follows. From equation [3-14], the potential for Bohm's non-relativistic model is the familiar expression:

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

and, as pointed out in equation [5-13], this potential can be written in the equivalent form:

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[1] de Broglie L., *Nonlinear Wave Mechanics*. Elsevier, Amsterdam (1960).

$$Q = \frac{1}{2m} \partial_j S \, \partial^j S - \partial_t S$$

In comparison with this last expression, the appropriate expression for de Broglie's relativistic model is:

$$Q = c \sqrt{(\partial_\mu S)\,(\partial^\mu S)} - mc^2 \qquad [7\text{-}4]$$

The corresponding equations of motion for the particle are then

$$\frac{d\mathbf{p}}{dt} = -\,\nabla Q$$

in the non-relativistic case and

$$\frac{dp_\mu}{d\tau} = \partial_\mu Q \qquad [7\text{-}5]$$

in the relativistic case ($\tau$ being the proper time).

There are three questionable features of de Broglie's relativistic model compared with Bohm's non-relativistic one:

1. De Broglie's model is based on the Klein-Gordon equation, whereas it might have been more appropriate to have a relativistic model corresponding to the Dirac equation. On the positive side, however, we are interested only in the non-relativistic limit and this limit is more easily derived in the Klein-Gordon case.
2. De Broglie bases his model on the Klein-Gordon equation's current density, which leads him to the following probability density for the particle's position at any time:

   $$\begin{aligned} P(\mathbf{x},t) &= \frac{i\hbar}{2mc^2} \left[\, \phi^* \,\partial_t\phi - \phi\,\partial_t\phi^* \,\right] \\ &= -\frac{1}{mc^2} R^2 \,\partial_t S \end{aligned} \qquad [7\text{-}6]$$

   This expression has the disadvantage of not being positive definite (unlike the simple expression $R^2$ in Bohm's model) and so requires the dubious notion of negative probabilities. De Broglie attempts to explain this result physically in terms of the particle's world line turning backwards in time. Fortunately this controversial point need not be considered further here because the probability density [7-6] reduces back to the positive expression $R^2$ in the non-relativistic limit.

3. In order for equations [7-3] and [7-5] to be compatible, de Broglie found it necessary to introduce a "variable" rest mass[2]:

$$\begin{aligned} M &= \frac{1}{c}\sqrt{(\partial_\mu S)\,(\partial^\mu S)} \\ &= m + \frac{Q}{c^2} \end{aligned} \qquad [7\text{-}7]$$

i.e., the rest mass is a function of the wavefunction $\phi$. Again, this rather unwelcome feature is no problem from our point of view because expression [7-7] can be shown to reduce back to the usual, constant mass m in the non-relativistic limit.

In Appendix 6 it is confirmed that equation [7-5] is consistent with [7-3] once [7-7] is assumed. In other words, we can assume in formulating our relativistic Lagrangian density that the particle's motion is still governed by a scalar potential.

## 7.2 Lagrangian Density for de Broglie's Model

By analogy with the Lagrangian density introduced in chapter 5 for Bohm's model, a similar expression will be proposed here for de Broglie's relativistic case. We will begin by simply stating the proposed expression, then discuss in detail the forms chosen for the various terms. Our relativistic Lagrangian density is:

$$\begin{aligned} å \; = \; & \frac{h^2}{2m}\left[\,(\partial_\mu \phi^*)\,(\partial^\mu \phi) - \left(\frac{mc}{h}\right)^2 \phi^* \phi\,\right] && (\textit{field terms}) \\ & - \rho_0\, mc \sqrt{u_\mu u^\mu} && (\textit{particle term}) \\ & - \rho_0\, Q \frac{\sqrt{u_\mu u^\mu}}{c} && (\textit{interaction term}) \end{aligned} \qquad [7\text{-}8]$$

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[2] In particular, de Broglie needed to relate the particle's 4-momentum to its 4-velocity via this factor.

where:

- $\phi$ is the Klein-Gordon wave-function,
- $\rho_0$ is the rest density distribution of the particle through space[3] ($\rho_0$ will be a delta function),
- m is the constant rest mass usually associated with the particle (not de Broglie's variable rest mass M),
- $u^\mu$ is the particle's 4-velocity ($u^\mu \equiv dx^\mu/d\tau$, where $\tau$ = proper time),
- Q is the scalar potential,
- c is the speed of light.

As with the Lagrangian densities considered in chapters 4 and 5, our expression here consists of a "free-field" component, a "particle" component and an "interaction" component. The free-field terms are the standard ones from which the Klein-Gordon equation may be derived[4]. The particle term and interaction terms are also standard expressions[5]. The form of this Lagrangian density is manifestly Lorentz invariant. The various constant factors in its terms ensure that it has the required dimensions of energy density.

It is to be understood here that, in deriving the equation of motion for the particle, one must employ the well-known technique of replacing the proper time with an arbitrary parameter while performing the variation process[6]. The interaction term in [7-8] is similar in appearance to the non-relativistic one in [5-1], except for the additional factor $\frac{\sqrt{u_\mu u^\mu}}{c}$, which ensures parameterisation independence of the action. This factor also ensures that

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[3] i.e., the matter density in the particle's instantaneous rest frame.
[4] See, e.g., p. 14 in Greiner W., *Relativistic Quantum Mechanics – Wave Equations*, 2nd Ed. Springer, Berlin (1997).
[5] See, e.g., p. 289 in Anderson J.L., *Principles of Relativity Physics*, Academic Press, N.Y. (1967).
[6] See, e.g., Sec. 7-9 in Goldstein H., *Classical Mechanics*, 2nd Ed. Addison-Wesley, Massachusetts (1980).

the equation of motion[7] obtained for the particle from this Lagrangian density is compatible with the familiar relativistic identity $u_\mu u^\mu = c^2$ that the 4-velocity must satisfy once the proper time is reinstated in place of the arbitrary parameter of variation. This point will be demonstrated in Appendix 7.

The identity $u_\mu u^\mu = c^2$ does, of course, allow us to rewrite the particle and interaction terms in the simpler forms

$$å_{\text{particle}} = -\rho_0\, mc^2 \qquad [7\text{-}9]$$

and

$$å_{\text{interaction}} = -\rho_0\, Q \qquad [7\text{-}10]$$

However, these forms are not suitable for obtaining the particle's equation of motion.

An explicit expression will be needed for the rest density distribution $\rho_0$ of the particle through space. We will take x to represent an arbitrary point in space-time and $x_0(\tau)$ to represent the particle's position in space-time at proper time $\tau$. The **rest** density is obtained by first noting (c.f., equation [5-3b]) that the particle's "ordinary" density distribution $\rho$ will be the delta function expression:

$$\begin{aligned} \rho &= \delta(x^1 - x_0^1)\,\delta(x^2 - x_0^2)\,\delta(x^3 - x_0^3) \\ &\equiv \delta(\mathbf{x} - \mathbf{x}_0) \end{aligned} \qquad [7\text{-}11]$$

where the boldface x's represent points in 3-space:

$$\mathbf{x} = (x^1, x^2, x^3)$$

Now, the two densities are connected by the following well-known relationship[8]:

$$\rho_0 = \frac{c}{u^0}\,\rho \qquad [7\text{-}12]$$

where $u^0$ is the time component of the particle's 4-velocity. Hence, combining [7-12] with [7-11], we have

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[7] See, e.g., p. 290 in Anderson J.L., *Principles of Relativity Physics*, Academic Press, N.Y. (1967).

$$\rho_0 = \frac{c}{u^0}\,\delta(\mathbf{x} - \mathbf{x}_0) \qquad [7\text{-}13]$$

This is the required expression for the particle's rest density distribution. It will be needed in the next section for the derivation of the particle's equation of motion.

Finally it should be noted that, in analogy to the non-relativistic case discussed in section 5.1, a more general relativistic Lagrangian density than [7-8] could be used in which the particle and interaction terms are multiplied by an arbitrary constant k. Again, this would leave the resulting equation of motion for the particle unchanged and would multiply the source term of the resulting field equation by k.

## 7.3 Equation of Motion for the Particle

In Appendix 7 it is confirmed that our proposed relativistic Lagrangian density yields the correct equation of motion [7-5]. Note that, as with the non-relativistic Lagrangian density in chapter 5, we are effectively treating the particle's velocity as an independent variable here and temporarily suspending the de Broglie-Bohm restriction $Mu^\mu = -\,\partial^\mu S$. This restriction can be restored at the end without any inconsistency once the Lagrangian formalism has yielded the required equations for energy and momentum conservation.

## 7.4 Field Equation

The field equation corresponding to the Lagrangian density [7-8] will now be considered. In analogy with the modified Schrodinger equation in chapter 5, this will be found to take the form of the Klein-Gordon equation with an extra term added. From equation [5-8], the appropriate form of Lagrange's equation for our needs is

$$\partial_\mu \frac{\partial \check{a}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \check{a}}{\partial \phi^*} = 0 \qquad [7\text{-}14]$$

[8] See, e.g., p. 122 in Rindler W., *Special Relativity*, 2nd Ed. Oliver and Boyd, Edinburgh (1969).

Now, there is no need to insert the field terms of [7-8] into equation [7-14], because that would simply yield the standard Klein-Gordon equation, as these terms have been designed to do[9]. Furthermore, inserting the particle term of [7-8] would simply yield zero, since this term is not a function of $\phi$. Therefore, anything extra to be added to the Klein-Gordon equation will come purely from the interaction term of the Lagrangian density. This term is

$$\partial\, å_{\text{interaction}} = -\rho_0\, Q\, \frac{\sqrt{u_\mu u^\mu}}{c}$$
$$= -\rho_0\, Q$$

and using the expression for the potential in equation [7-4], it can be written as:

$$å_{\text{interaction}} = -\{c\sqrt{(\partial_\mu S)\,(\partial^\mu S)} - mc^2\}\,\rho_0 \qquad [7\text{-}15]$$

The additional term that arises when this interaction term is inserted into [7-14] is derived in Appendix 8. The result is that the following “source term” is obtained:

$$\frac{i\hbar c}{2}\,\frac{1}{\phi^*}\,\partial_\mu \frac{\rho_0\,\partial^\mu S}{\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}} \qquad [7\text{-}16]$$

so that the Klein-Gordon equation [7-1] is modified to:

$$\frac{\hbar^2}{2m}\,[\partial_\mu\partial^\mu\phi + (\frac{mc}{\hbar})^2\phi] \;=\; \frac{i\hbar c}{2}\,\frac{1}{\phi^*}\,\partial_\mu \frac{\rho_0\,\partial^\mu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} \qquad [7\text{-}17]$$

Once the restriction $p^\mu = -\,\partial^\mu S$ is reimposed, this equation can be simplified (with the aid of [7-7]) to:

$$\frac{\hbar^2}{2m}\,[\partial_\mu\partial^\mu\phi + (\frac{mc}{\hbar})^2\phi] \;=\; -\,\frac{i\hbar}{2}\,\frac{1}{\phi^*}\,\partial_\mu \frac{\rho_0\, p^\mu}{M}$$

i.e.,

$$\frac{\hbar^2}{2m}\,[\partial_\mu\partial^\mu\phi + (\frac{mc}{\hbar})^2\phi] \;=\; -\,\frac{i\hbar}{2}\,\frac{1}{\phi^*}\,\partial_\mu(\rho_0 u^\mu) \qquad [7\text{-}18]$$

[9] See, e.g., p.14 in Greiner W., *Relativistic Quantum Mechanics – Wave Equations*, 2nd Ed. Springer, Berlin (1997).

In analogy with the modified Schrodinger equation in chapter 5, the expression $\partial_\mu(\rho_0 u^\mu)$ on the right of [7-18] is seen to resemble the form of a continuity equation. This expression equalling zero is, in fact, the condition for conservation of the "matter" making up the particle. In the low energy, single particle case (i.e., in the absence of creation and annihilation), this expression is zero and our new field equation simply reduces back to the standard Klein-Gordon equation.

### 7.5 Energy-Momentum Tensor for the Particle

The remainder of this chapter will be concerned with demonstrating conservation of energy and momentum for the relativistic model under consideration here. This will be achieved by considering the energy-momentum tensors corresponding to the various terms in the Lagrangian density [7-8]. The main result will be derived in the next section. As a preliminary step, we will briefly focus on the energy-momentum tensor for the particle. The expression for this tensor has already been given in equation [6-16]. Allowing for the variable rest mass M in de Broglie's model, the particle's energy-momentum tensor has the form:

$$T^{\mu\nu}_{particle} = \rho_0 M u^\mu u^\nu \qquad [7\text{-}19]$$

where $u^\mu$ is the particle's 4-velocity and the rest density $\rho_0$ is defined in [7-13]. It is a standard result[10] that the divergence of this tensor is related to the rate of change of the particle's 4-momentum as follows:

$$\partial_\nu T^{\mu\nu}_{particle} = \rho_0 \frac{dp^\mu}{d\tau} \qquad [7\text{-}20]$$

Combining this with the equation of motion [7-5]:

$$\frac{dp_\mu}{d\tau} = \partial_\mu Q$$

we then obtain the relationship:

$\partial_\nu T^{\mu\nu}_{\text{particle}} = \rho_0\, \partial^\mu Q$ [7-21]

This result will be needed in the following section.

## 7.6 Noether's Theorem adapted to the Present Case

A formulation of Noether's theorem designed specifically to serve our particular needs will now be developed from first principles. As discussed at the end of the previous chapter, this way of proceeding is necessary because of difficulties that arise in attempting a more routine approach.

In most textbook examples of classical particle-field interactions, the **interaction** term of the Lagrangian density does not involve derivatives of the field. This can be shown to have the consequence that the overall energy-momentum tensor for that Lagrangian density consists simply of $T^{\mu\nu}{}_{\text{field}}$ plus $T^{\mu\nu}{}_{\text{particle}}$, with no additional terms $T^{\mu\nu}{}_{\text{interaction}}$. For our more complex interaction term [7-15], this simple situation no longer holds. To find the more general expression for the overall $T^{\mu\nu}$ that is applicable to our case, we will return to Noether's theorem and derive the required expression.

Our Lagrangian density is an explicit function of the field, its first derivatives and the particle's rest density, rest mass and 4-velocity:

å = å $(\phi, \phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0, m, u^\alpha)$ [7-22]

The dependence of å on the particle's rest mass and 4-velocity will turn out to be irrelevant here, because m is a constant and $u^\alpha$ is not an explicit function of the coordinates x (it is actually a function of the proper time $\tau$). Equation [7-22] can therefore be written more conveniently for our purposes as:

å = å $(\phi, \phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0)$ [7-23]

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[10] See, e.g., pp. 25-27 in Felsager B., *Geometry, Particles and Fields*. Springer, N.Y., (1998).

Noether's theorem states that the system's energy and momentum will be conserved provided å is not an explicit function of the coordinates, i.e., provided we are **not** required to write:

$$\mathring{a} = \mathring{a}\,(\phi, \phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0, x) \qquad [7\text{-}24]$$

We now take the partial derivative of å with respect to $x_\mu$, holding the other three coordinates $x_\lambda$ constant, where $\lambda \neq \mu$. From [7-23], the full expression for this derivative is

$$\begin{aligned}[\frac{\partial\,\mathring{a}}{\partial x_\mu}]_{x_\lambda} = {} & [\frac{\partial\,\mathring{a}}{\partial\phi}]_{\phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0}\,[\frac{\partial\phi}{\partial x_\mu}]_{x_\lambda} + [\frac{\partial\,\mathring{a}}{\partial\phi^*}]_{\phi, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0}\,[\frac{\partial\phi^*}{\partial x_\mu}]_{x_\lambda} \\ & + [\frac{\partial\,\mathring{a}}{\partial(\partial_\nu\phi)}]_{\phi, \phi^*, \partial_\alpha\phi^*, \rho_0}\,[\frac{\partial(\partial_\nu\phi)}{\partial x_\mu}]_{x_\lambda} + [\frac{\partial\,\mathring{a}}{\partial(\partial_\nu\phi^*)}]_{\phi, \phi^*, \partial_\alpha\phi, \rho_0}\,[\frac{\partial(\partial_\nu\phi^*)}{\partial x_\mu}]_{x_\lambda} \qquad [7\text{-}25a] \\ & + [\frac{\partial\,\mathring{a}}{\partial\,\rho_0}]_{\phi, \phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*}\,[\frac{\partial\rho_0}{\partial x_\mu}]_{x_\lambda}\end{aligned}$$

where the quantities held constant in each partial differentiation have been explicitly displayed outside the square brackets. The third and fourth terms on the right each contain a summation over $\nu$. Note that in the more general case of [7-24] (instead of [7-23]), the right-hand side of [7-25a] would have the extra term:

$$[\frac{\partial\,\breve{a}}{\partial x_\mu}]_{\phi, \phi^*, \partial_\alpha\phi, \partial_\alpha\phi^*, \rho_0, x_\lambda}$$

since this term would then no longer be zero. (Also note how this term differs from the one on the left-hand side of [7-25a].) From here on, the quantities held constant will not be shown. Equation [7-25a] is then written more simply as:

$$\frac{\partial\,\breve{a}}{\partial x_\mu} = \frac{\partial\,\breve{a}}{\partial\phi}\,\partial^\mu\phi + \frac{\partial\,\breve{a}}{\partial\phi^*}\,\partial^\mu\phi^* + \frac{\partial\,\breve{a}}{\partial(\partial_\nu\phi)}\,\partial^\mu(\partial_\nu\phi) + \frac{\partial\,\breve{a}}{\partial(\partial_\nu\phi^*)}\,\partial^\mu(\partial_\nu\phi^*) + \frac{\partial\,\breve{a}}{\partial\rho_0}\,\partial^\mu\rho_0 \qquad [7\text{-}25b]$$

Now the first and second terms on the right of [7-25b] can be modified using the field equation [7-14] for $\phi$ and the complex conjugate equation for $\phi^*$:

$$\partial_\mu\,\frac{\partial\,\breve{a}}{\partial(\partial_\mu\phi^*)} - \frac{\partial\,\breve{a}}{\partial\phi^*} = 0$$

$$\partial_\mu \frac{\partial \breve{a}}{\partial(\partial_\mu \phi)} - \frac{\partial \breve{a}}{\partial \phi} = 0$$

Equation [7-25b] then becomes

$$\frac{\partial \mathring{a}}{\partial x_\mu} = [\partial_\nu \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)}]\, \partial^\mu \phi + [\partial_\nu \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)}]\, \partial^\mu \phi^* + \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)}\, \partial^\mu \partial_\nu \phi + \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)}\, \partial^\mu \partial_\nu \phi^* + \frac{\partial \mathring{a}}{\partial \rho_0}\, \partial^\mu \rho_0$$

$$= (\partial^\mu \phi)\, \partial_\nu \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \partial_\nu \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} + (\partial_\nu \partial^\mu \phi)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial_\nu \partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} + \frac{\partial \mathring{a}}{\partial \rho_0}\, \partial^\mu \rho_0$$

$$= \partial_\nu \left\{ (\partial^\mu \phi)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} \right\} + \frac{\partial \mathring{a}}{\partial \rho_0}\, \partial^\mu \rho_0$$

This equation can be rearranged to

$$\partial_\nu \left\{ (\partial^\mu \phi)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} \right\} - \frac{\partial \mathring{a}}{\partial x_\mu} + \frac{\partial \mathring{a}}{\partial \rho_0}\, \partial^\mu \rho_0 = 0$$

or, equivalently

$$\partial_\nu \left\{ (\partial^\mu \phi)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu}\, \mathring{a} \right\} + \frac{\partial \mathring{a}}{\partial \rho_0}\, \partial^\mu \rho_0 = 0 \qquad [7\text{-}26]$$

In the special case when the Lagrangian density does **not** contain $\rho_0$, i.e., there are no particles present, equation [7-26] reduces to

$$\partial_\nu \left\{ (\partial^\mu \phi\,)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu}\, \mathring{a} \right\} = 0 \qquad [7\text{-}27]$$

The curly bracket in [7-27] is then seen to have zero divergence and we can identify it as $T^{\mu\nu}$, this definition being in agreement with [6-23] earlier. Energy and momentum conservation is thereby achieved for this case.

To deal with the more general case where $\rho_0$ is not zero, we will return to [7-26] and rewrite it as

$$\partial_\nu \left\{ (\partial^\mu \phi)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*)\, \frac{\partial \mathring{a}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu}\, \mathring{a} \right\} + \partial^\mu \left( \frac{\partial \mathring{a}}{\partial \rho_0}\, \rho_0 \right) - \rho_0\, \partial^\mu \left( \frac{\partial \mathring{a}}{\partial \rho_0} \right) = 0$$

which then gives us

$$\partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu} \left( \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \rho_0} \rho_0 \right) \right\} - \rho_0 \, \partial^\mu \left( \frac{\partial \mathcal{L}}{\partial \rho_0} \right) = 0$$

Now, for the particular case of our Lagrangian density [7-8], this becomes

$$\partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu} \left( \mathcal{L} - \mathcal{L}_{\text{particle}} - \mathcal{L}_{\text{interaction}} \right) \right\}$$
$$- \rho_0 \, \partial^\mu \left( - mc \sqrt{u_\mu u^\mu} - Q \frac{\sqrt{u_\mu u^\mu}}{c} \right) = 0$$

i.e.,

$$\partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu} \, \mathcal{L}_{\text{field}} \right\} + \rho_0 \, \partial^\mu Q = 0$$

and making use of the relationship [7-21] concerning the divergence of $T^{\mu\nu}{}_{\text{particle}}$, we obtain

$$\partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu} \, \mathcal{L}_{\text{field}} \right\} + \partial_\nu \, T^{\mu\nu}_{\text{particle}} = 0 \qquad [7\text{-}28]$$

Noting that $\mathcal{L}_{\text{particle}}$ is not a function of $\partial_\nu \phi$ or $\partial_\nu \phi^*$, [7-28] can be written as

$$\partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\nu \phi^*)} - g^{\mu\nu} \, \mathcal{L}_{\text{field}} \right\}$$
$$+ \partial_\nu \left\{ (\partial^\mu \phi) \frac{\partial \mathcal{L}_{\text{interaction}}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \frac{\partial \mathcal{L}_{\text{interaction}}}{\partial(\partial_\nu \phi^*)} \right\} \qquad [7\text{-}29]$$
$$+ \partial_\nu \, T^{\mu\nu}_{\text{particle}} = 0$$

The first curly bracket in [7-29] can be recognised as $T^{\mu\nu}{}_{\text{field}}$ (c.f., expression [6-23]). Introducing the definition

$$T^{\mu\nu}_{\text{interaction}} \equiv (\partial^\mu \phi) \frac{\partial \mathcal{L}_{\text{interaction}}}{\partial(\partial_\nu \phi)} + (\partial^\mu \phi^*) \left( \frac{\partial \mathcal{L}_{\text{interaction}}}{\partial(\partial_\nu \phi^*)} \right)$$

equation [7-29] can therefore be expressed as

$$\partial_\nu \, T^{\mu\nu}_{\text{field}} + \partial_\nu \, T^{\mu\nu}_{\text{interaction}} + \partial_\nu \, T^{\mu\nu}_{\text{particle}} = 0$$

## 7.7 Summary of Equations describing Overall Conservation

Summarising the results of the previous section, conservation of energy and momentum for the Lagrangian density [7-8] is described by the condition:

$$\partial_\nu \, T^{\mu\nu}_{total} = 0 \qquad [7\text{-}30]$$

where the overall energy-momentum tensor can be written component-wise as:

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{field} + T^{\mu\nu}_{particle} + T^{\mu\nu}_{interaction} \qquad [7\text{-}31]$$

and the individual tensor components have been determined to be:

$$T^{\mu\nu}_{field} = [\, \partial^\mu\phi \frac{\partial}{\partial(\partial_\nu\phi)} + \partial^\mu\phi^* \frac{\partial}{\partial(\partial_\nu\phi^*)} - g^{\mu\nu} \,] \text{ å }_{field} \qquad [7\text{-}32]$$

$$T^{\mu\nu}_{particle} = \rho_0 M u^\mu u^\nu \qquad [7\text{-}33]$$

$$T^{\mu\nu}_{interaction} = [\, \partial^\mu\phi \frac{\partial}{\partial(\partial_\nu\phi)} + \partial^\mu\phi^* \frac{\partial}{\partial(\partial_\nu\phi^*)} \,] \text{ å }_{interaction} \qquad [7\text{-}34]$$

Note that $T^{\mu\nu}_{interaction}$ is zero when $\mathcal{L}_{interaction}$ does not involve derivatives of the field. This is actually the case for the electromagnetic field and for most classical examples given in textbooks. It is not the case, however, for our $\mathcal{L}_{interaction}$.

## 7.8 Energy-Momentum Tensors $T^{\mu\nu}_{field}$ and $T^{\mu\nu}_{interaction}$

Equation [7-33] above gives the explicit expression for $T^{\mu\nu}_{particle}$ for any relativistic Lagrangian density. Explicit expressions for $T^{\mu\nu}_{field}$ and $T^{\mu\nu}_{interaction}$ for our particular Lagrangian density will now be considered. The expression for $T^{\mu\nu}_{field}$ in the case of a free Klein-Gordon field is well known[11], so there is no need to derive it here from equation [7-32]. It has the form:

$$T^{\mu\nu}_{field} = \frac{\hbar^2}{2m} \{(\partial^\mu\phi)(\partial^\nu\phi^*) + (\partial^\mu\phi^*)(\partial^\nu\phi) - g^{\mu\nu} [(\partial_\lambda\phi^*)(\partial^\lambda\phi) - (\frac{mc}{\hbar})^2 \phi^*\phi]\} \qquad [7\text{-}35]$$

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[11] See, e.g., p. 15 in Greiner W., *Relativistic Quantum Mechanics – Wave Equations*, 2nd Ed. Springer, Berlin (1997).

It therefore remains for us to find an explicit expression only for $T^{\mu\nu}{}_{interaction}$. To do this, we need to evaluate [7-34] for our particular interaction term [7-15]:

$$\mathcal{L}_{interaction} = -\{c\sqrt{(\partial_\mu S)(\partial^\mu S)} - mc^2\}\,\rho_0 \qquad [7\text{-}36]$$

Inserting [7-36] into [7-34] yields:

$$\begin{aligned}
T^{\mu\nu}_{interaction} &= (\partial^\mu\phi)\,\frac{\partial[-\{c\sqrt{(\partial_\alpha S)(\partial^\alpha S)} - mc^2\}\,\rho_0]}{\partial(\partial_\nu\phi)} + (\partial^\mu\phi^*)\,\frac{\partial[-\{c\sqrt{(\partial_\alpha S)(\partial^\alpha S)} - mc^2\}\,\rho_0]}{\partial(\partial_\nu\phi^*)} \\
&= -c\,\{(\partial^\mu\phi)\,\frac{1}{2\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\frac{\partial[(\partial_\lambda S)(\partial^\lambda S)]}{\partial(\partial_\nu\phi)} + (\partial^\mu\phi^*)\,\frac{1}{2\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\frac{\partial[(\partial_\lambda S)(\partial^\lambda S)]}{\partial(\partial_\nu\phi^*)}\}\,\rho_0 \\
&= -\frac{c}{\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\{(\partial^\mu\phi)(\partial_\lambda S)\,\frac{\partial(\partial^\lambda S)}{\partial(\partial_\nu\phi)} + (\partial^\mu\phi^*)(\partial_\lambda S)\,\frac{\partial(\partial^\lambda S)}{\partial(\partial_\nu\phi^*)}\}\,\rho_0
\end{aligned}$$

[7-37]

Now, using the fact that $(\partial^\nu S)$ can be written in terms of the wavefunction and its complex conjugate as follows:

$$\partial_\mu S = -\frac{i\hbar}{2}\,\{\frac{\partial_\mu\phi}{\phi} - \frac{\partial_\mu\phi^*}{\phi^*}\} \qquad [7\text{-}38]$$

we obtain the identities:

$$\frac{\partial(\partial^\lambda S)}{\partial(\partial_\nu\phi)} = -\frac{i\hbar}{2}\,\{\frac{g^{\lambda\nu}}{\phi} - 0\}$$

and

$$\frac{\partial(\partial^\lambda S)}{\partial(\partial_\nu\phi^*)} = -\frac{i\hbar}{2}\,\{0 - \frac{g^{\lambda\nu}}{\phi^*}\}$$

Hence [7-37] becomes

$$\begin{aligned}
T^{\mu\nu}_{interaction} &= \frac{i\hbar c}{2\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\{(\partial^\mu\phi)(\partial_\lambda S)\,\frac{g^{\lambda\nu}}{\phi} - (\partial^\mu\phi^*)(\partial_\lambda S)\,\frac{g^{\lambda\nu}}{\phi^*}\}\,\rho_0 \\
&= \frac{i\hbar c}{2\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\{(\partial^\mu\phi)(\partial^\nu S)\,\frac{1}{\phi} - (\partial^\mu\phi^*)(\partial^\nu S)\,\frac{1}{\phi^*}\}\,\rho_0 \\
&= \frac{i\hbar c}{2\sqrt{(\partial_\alpha S)(\partial^\alpha S)}}\,\{\frac{\partial^\mu\phi}{\phi} - \frac{\partial^\mu\phi^*}{\phi^*}\}\,(\partial^\nu S)\,\rho_0
\end{aligned}$$

and using [7-38] we finally obtain

$$T^{\mu\nu}_{interaction} = \frac{-c\,(\partial^\mu S)(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\alpha S)(\partial^\alpha S)}} \qquad [7\text{-}39]$$

This is the desired term needed to complete the system's overall energy-momentum tensor. (It should be kept in mind that it is possible to write out this expression in full using $\phi$ and $\phi^*$ only, instead of using S as an abbreviation.)

In summary, gathering [7-33], [7-35] and [7-39] together, the overall energy-momentum tensor for the system described by our relativistic Lagrangian density is made up of the following parts:

$$T^{\mu\nu}_{\text{field}} = \frac{\hbar^2}{2m} \{(\partial^\mu\phi)(\partial^\nu\phi^*) + (\partial^\mu\phi^*)(\partial^\nu\phi) - g^{\mu\nu} [(\partial_\lambda\phi^*)(\partial^\lambda\phi) - (\tfrac{mc}{\hbar})^2 \phi^*\phi]\} \qquad [7\text{-}40]$$

$$T^{\mu\nu}_{\text{particle}} = \rho_0 M u^\mu u^\nu \qquad [7\text{-}41]$$

$$T^{\mu\nu}_{\text{interaction}} = \frac{-c\,(\partial^\mu S)\,(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\alpha S)\,(\partial^\alpha S)}} \qquad [7\text{-}42]$$

## 7.9 Divergence and Conservation

The final task in this chapter is to check explicitly that the divergence of the overall energy-momentum tensor for the particle-field system is zero and thereby confirm that energy and momentum are conserved. Towards this end, the divergences of $T^{\mu\nu}{}_{\text{field}}$, $T^{\mu\nu}{}_{\text{particle}}$ and $T^{\mu\nu}{}_{\text{interaction}}$ will be evaluated separately.

### 7.9.1 Divergence of $T^{\mu\nu}{}_{\text{field}}$

Taking the divergence of expression [7-40], we obtain

$$\begin{aligned}
\partial_\nu T^{\mu\nu}_{\text{field}} &= \frac{\hbar^2}{2m} \{(\partial_\nu\partial^\mu\phi)(\partial^\nu\phi^*) + (\partial^\mu\phi)(\partial_\nu\partial^\nu\phi^*) + (\partial_\nu\partial^\mu\phi^*)(\partial^\nu\phi) + (\partial^\mu\phi^*)(\partial_\nu\partial^\nu\phi) \\
&\qquad - \partial^\mu [(\partial_\lambda\phi^*)(\partial^\lambda\phi) - (\tfrac{mc}{\hbar})^2\phi^*\phi]\} \\
&= \frac{\hbar^2}{2m} \{(\partial_\lambda\partial^\mu\phi)(\partial^\lambda\phi^*) + (\partial^\mu\phi)(\partial_\nu\partial^\nu\phi^*) + (\partial_\lambda\partial^\mu\phi^*)(\partial^\lambda\phi) + (\partial^\mu\phi^*)(\partial_\nu\partial^\nu\phi) \\
&\qquad - (\partial^\mu\partial_\lambda\phi^*)(\partial^\lambda\phi) - (\partial_\lambda\phi^*)(\partial^\mu\partial^\lambda\phi) + \partial^\mu[(\tfrac{mc}{\hbar})^2 \phi^*\phi]\} \\
&= \frac{\hbar^2}{2m} \{(\partial^\mu\phi)(\partial_\nu\partial^\nu\phi^*) + (\partial^\mu\phi^*)(\partial_\nu\partial^\nu\phi) + (\tfrac{mc}{\hbar})^2(\phi^*\,\partial^\mu\phi + \phi\,\partial^\mu\phi^*)\}
\end{aligned}$$

[7-43]

This can be simplified further by using the field equation corresponding to our Lagrangian density, i.e., by using the extended Klein-Gordon equation [7-17]:

$$\frac{\hbar^2}{2m}\,[\partial_\mu\partial^\mu\phi + (\frac{mc}{\hbar})^2\phi] \;=\; \frac{i\hbar c}{2}\,\frac{1}{\phi^*}\,\partial_\mu\,\frac{\rho_0\,\partial^\mu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}$$

which can be written more conveniently in the form:

$$\partial_\nu\partial^\nu\phi \;=\; -\,(\frac{mc}{\hbar})^2\phi + \frac{imc}{\hbar}\,\frac{1}{\phi^*}\,\partial_\nu\,\frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} \qquad [7\text{-}44]$$

Inserting [7-44] and its complex conjugate into [7-43], we obtain

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\text{field}} = \frac{\hbar^2}{2m}\,\{&(\partial^\mu\phi)\,[\,-\,(\frac{mc}{\hbar})^2\,\phi^* - \frac{imc}{\hbar}\,\frac{1}{\phi}\,\partial_\nu\,\frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}\,]\\ &+ (\partial^\mu\phi^*)\,[\,-\,(\frac{mc}{\hbar})^2\,\phi + \frac{imc}{\hbar}\,\frac{1}{\phi^*}\,\partial_\nu\,\frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}\,]\\ &+ (\frac{mc}{\hbar})^2(\phi^*\,\partial^\mu\phi + \phi\,\partial^\mu\phi^*)\}\\ = -\,\frac{i\hbar c}{2}\,&[\frac{\partial^\mu\phi}{\phi} - \frac{\partial^\mu\phi^*}{\phi^*}]\,\partial_\nu\,\frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}\end{aligned}$$

and using [7-38]:

$$\partial_\mu S = -\,\frac{i\hbar}{2}\,\{\frac{\partial_\mu\phi}{\phi} - \frac{\partial_\mu\phi^*}{\phi^*}\}$$

the divergence of $T^{\mu\nu}{}_{\text{field}}$ is seen to reduce to

$$\partial_\nu T^{\mu\nu}_{\text{field}} = c\,(\partial^\mu S)\,\partial_\nu\,\frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} \qquad [7\text{-}45]$$

### 7.9.2 Divergence of $T^{\mu\nu}{}_{\text{particle}}$

The divergence of $T^{\mu\nu}{}_{\text{particle}}$ for the particular case of our Lagrangian density has already been stated earlier. From [7-21], it is:

$$\partial_\nu T^{\mu\nu}_{\text{particle}} = \rho_0\,\partial^\mu Q \qquad [7\text{-}46]$$

### 7.9.3 Divergence of $T^{\mu\nu}{}_{\text{interaction}}$

Taking the divergence of expression [7-42], we obtain

$$\partial_\nu T^{\mu\nu}_{interaction} = -c\,(\partial^\mu S)\,\partial_\nu \frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} - c\,\frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}\,\partial_\nu(\partial^\mu S)$$

$$= -c\,(\partial^\mu S)\,\partial_\nu \frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} - c\,\frac{\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}}\,(\partial^\nu S)\partial^\mu(\partial_\nu S)$$

$$= -c\,(\partial^\mu S)\,\partial_\nu \frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} - \rho_0\partial^\mu\,(c\sqrt{(\partial_\nu S)(\partial^\nu S)}$$

and using expression [7-4] for the quantum potential:

$$Q = c\,\sqrt{(\partial_\mu S)\,(\partial^\mu S)} - mc^2$$

the divergence of $T^{\mu\nu}{}_{interaction}$ then becomes

$$\partial_\nu T^{\mu\nu}_{interaction} = -c\,(\partial^\mu S)\,\partial_\nu \frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} - \rho_0\,\partial^\mu Q \qquad [7\text{-}47]$$

### 7.9.4 Divergence of $T^{\mu\nu}{}_{total}$

From equation [7-31] we have:

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{field} + T^{\mu\nu}_{particle} + T^{\mu\nu}_{interaction}$$

The divergence of this overall energy-momentum tensor can now be obtained by combining [7-45], [7-46] and [7-47] to obtain:

$$\partial_\nu T^{\mu\nu}_{total} = \partial_\nu T^{\mu\nu}_{field} + \partial_\nu T^{\mu\nu}_{particle} + \partial_\nu T^{\mu\nu}_{interaction}$$

$$= \left\{ c\,(\partial^\mu S)\,\partial_\nu \frac{\rho_0\,\partial^\nu S}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} \right\} + \left\{ \rho_0\,\partial^\mu Q \right\} + \left\{ -c\,(\partial^\mu S)\,\partial_\nu \frac{(\partial^\nu S)\,\rho_0}{\sqrt{(\partial_\lambda S)(\partial^\lambda S)}} - \rho_0\,\partial^\mu Q \right\}$$

which cancels to:

$$\partial_\nu T^{\mu\nu}_{total} = 0$$

This is the desired result for energy and momentum conservation. (The divergence calculation above also serves as a useful double-check on our derivations of $T^{\mu\nu}{}_{field}$, $T^{\mu\nu}{}_{particle}$ and $T^{\mu\nu}{}_{interaction}$.)

Therefore, from the viewpoint of conservation, a satisfactory relativistic model has been achieved.