Chapter 3: Bohm's Model

3.1 Summary of Bohm's Model

For non-relativistic quantum mechanics, David Bohm has explicitly constructed a scheme which supports a continuously evolving underlying "particle trajectory" and yields results entirely consistent with experimental evidence[1]. Even if suitable for no other purpose, the Bohm model has demonstrated that an unqualified refutation of hidden variables theories is, in fact, not possible. This model also refutes certain other claims, such as that we must necessarily abandon realism, determinism, analyzability, etc.

The mathematical structure of the Bohmian model arises from combining the Schrodinger Equation, the Equation of Continuity and the requirement of Conservation of Probability in a fairly straightforward manner. Writing the wavefunction in the form:

$$\psi(\mathbf{x},t) = R(\mathbf{x},t) \exp\left(\frac{iS(\mathbf{x},t)}{\hbar}\right) \qquad [3\text{-}1]$$

Bohm's non-relativistic model requires three basic physical assumptions:

1. An electron or other quantum entity is a particle (represented by a position coordinate $\mathbf{x}$ that is a well-defined, continuous function of time).
2. The particle's velocity is given at all times by $\mathbf{v} = \nabla S/m$.
3. $P(\mathbf{x};t) = R^2$ is the probability distribution for particle positions in a statistical ensemble of similar systems.

[1] Bohm. D., *A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables*, Physical Review Vol. 85, pp. 166-179 and 180- 193 (1952).
Bohm D. and Hiley B.J., *Measurement Understood Through the Quantum Potential Approach,* Foundations of Physics Vol. 14, pp. 254-274 (1984).

3.1.1 Equation of Continuity

The Schrodinger Equation[2] and its complex conjugate can be written as:

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x})\,\psi = i\hbar\,\frac{\partial\psi}{\partial t} \qquad [3\text{-}2a]$$

$$\frac{-\hbar^2}{2m}\nabla^2\psi^* + V(\mathbf{x})\,\psi^* = -\,i\hbar\,\frac{\partial\psi^*}{\partial t} \qquad [3\text{-}2b]$$

and the classical equation of continuity for fluids takes the form[3]:

$$\nabla\,.\,\mathbf{j} + \frac{\partial\rho}{\partial t} = 0\;;\quad \mathbf{j} = \rho\,\mathbf{v} \qquad [3\text{-}3]$$

Here, $\mathbf{j}(\mathbf{x},t)$ is the fluid flux, or mass of fluid passing through a defined unit cross-section per unit time. The flux is obtained by multiplying the flow velocity $\mathbf{v}(\mathbf{x},t)$ by the fluid's local density $\rho(\mathbf{x},t)$ within the cross-section.

Evaluating Ψ^* x (SE) - Ψ x (SE)*, where SE denotes the Schrodinger equation, the following expression may be obtained[4]:

$$-\frac{\hbar^2}{2m}\nabla(\,\psi^*\nabla\psi - \psi\,\nabla\psi^*\,) - i\hbar\,\frac{\partial(\psi^*\psi)}{\partial t} = 0 \qquad [3\text{-}4]$$

Using the R,S polar notation for the complex function Ψ, this equation reduces to:

$$\nabla\left[\,\frac{R^2\,\nabla S}{m}\,\right] + \frac{\partial\,R^2}{\partial t} = 0 \qquad [3\text{-}5]$$

Comparing equations [3-3] and [3-5], Bohm's Model develops from making the obvious associations:

$$\rho = R^2 \qquad [3\text{-}6]$$

$$\mathbf{v} = \nabla S/m \qquad [3\text{-}7]$$

The second of these equations can be rewritten as a momentum expression:

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[2] See, e.g., p. 95 in Saxon D.S., *Elementary Quantum Mechanics*. Holden Day Publishers, San Francisco, California. (1968).

[3] See, e.g., p.121 in Messiah A., *Quantum Mechanics.* Vol. 1. North - Holland Publishing Company Amsterdam (1964).

$\mathbf{p} = \nabla S$ [3-8]

The minimalist version of Bohm's model requires only the three basic physical assumptions numbered above. However, the original presentation of Bohm's model, which may be distinguished as the "de Broglie-Bohm model," included a derived quantum potential Q, outlined in the next section. It was later realised that Bohm's model did not actually require presentation of the quantum potential to reach agreement with experimental results. For this reason the inclusion of the quantum potential Q is not actually necessary. Durr, Goldstein and Zanghi have stated that, from their perspective, the artificiality suggested by the quantum potential is the price one pays for attempting to cast the non-classical Bohmian theory into a classical mould[5]. They use the name "Bohmian mechanics" for the minimalist version of the theory which does not contain the quantum potential in its formulation.

In Bohm's model, the use of statistics via $P(\mathbf{x},t) = R^2(\mathbf{x},t)$ is a consequence only of our ignorance of the particles exact position rather than being inherent in the conceptual structure of the model. The wave function ψ plays two conceptually distinct roles in that it determines both the influence of the environment on the particle's position[6] **and** the probability density $P(\mathbf{x},t)$ [7].

[4] See, e.g., pp. 25-27 in Schiff L.I., *Quantum Mechanics*, 3rd Edition. McGraw Hill Book Company (1968).
[5] Cushing J.T., *Quantum Mechanics: Historical Contingency & the Copenhagen Hegemony.* p. 45. University of Chicago Press (1994). (See also other references cited therein.)
[6] More fundamentally, the wave function generates the vector field on configuration space defining the equation of motion of the particle.
[7] Durr D., Goldstein S. and Zanghi N., *Quantum mechanics, Randomness, and Deterministic Reality.* Physics Letters A. Vol. 172, pp. 6-12 (1992). See also footnote 5 above.

3.1.2 Hamiltonian - Energy Considerations

By evaluating $\Psi^* \times (SE) + \Psi \times (SE)^*$, it is possible to produce an equation containing terms similar to the classical Hamiltonian, a function which expresses the system's energy in terms of momentum **p**, position **x** and possibly the time t. The relevant classical equation is:

Kinetic Energy + Potential Energy = Total Energy [3-9]

which can be written as[8]:

$$\frac{p^2}{2m} + PE = E \qquad [3\text{-}10]$$

In the case of Bohm's model, the corresponding equation is[9]:

$$\frac{[\nabla S]^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = -\frac{\partial S}{\partial t} \qquad [3\text{-}11]$$

Given the previous association $\mathbf{p} = \boldsymbol{\nabla} S$ for momentum, the Schrodinger equation can now be reinterpreted, within Bohm's model, as representing a classical particle having potential energy and total energy given, respectively, by[10]:

$$PE = V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \qquad [3\text{-}12]$$

$$E = -\frac{\partial S}{\partial t} \qquad [3\text{-}13]$$

The potential consists of a classical component V plus a quantum component, usually represented by the letter Q:

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \qquad [3\text{-}14]$$

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[8] Here, p^2 is taken to mean $|p|^2$ which is simply $\mathbf{p} \,.\, \mathbf{p}$ (similarly $[\nabla S]^2 = |\nabla S|^2 = \boldsymbol{\nabla} S \,.\, \boldsymbol{\nabla} S$).
[9] It is assumed in this thesis that $V(\mathbf{x})$ is real.
[10] Bohm. D., *A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables*, Part1, Physical Review Vol. 85, pp. 166-179 (1952).

3.1.3 Potential Gradient and Force in Bohm's Model

It can be shown[11] that the "quantum mechanical force" required to produce the accelerations described implicitly by the velocity relationship $\mathbf{v} = \nabla S/m$ is equal to minus the gradient of the potential given in [3-14]. The derivation is as follows (i and j have the values 1,2 and 3 here, x^i and x_i are related via $x^i = -x_i$ and a summation is implied over repeated indices):

$$F^i = \frac{d}{dt}(mv^i) \qquad [3\text{-}15a]$$

$$= m\left(\frac{dx^j}{dt}\frac{\partial v^i}{\partial x^j} + \frac{dt}{dt}\frac{\partial v^i}{\partial t}\right) \qquad [3\text{-}15b]$$

$$= \left(v^j\frac{\partial}{\partial x^j} + \frac{\partial}{\partial t}\right) mv^i \qquad [3\text{-}15c]$$

Substituting in the expression $mv^i = -\,\partial S/\partial x_i$ from equation [3-7], we obtain:

$$F^i = -\left(-\frac{1}{m}\frac{\partial S}{\partial x_j}\frac{\partial}{\partial x^j} + \frac{\partial}{\partial t}\right)\frac{\partial S}{\partial x_i} \qquad [3\text{-}16a]$$

$$= -\left(-\frac{1}{m}\frac{\partial S}{\partial x_j}\frac{\partial^2 S}{\partial x^j \partial x_i} + \frac{\partial^2 S}{\partial t \partial x_i}\right) \qquad [3\text{-}16b]$$

$$= -\frac{\partial}{\partial x_i}\left(-\frac{1}{2m}\frac{\partial S}{\partial x_j}\frac{\partial S}{\partial x^j} + \frac{\partial S}{\partial t}\right) \qquad [3\text{-}16c]$$

and using the relationship $x_i = -\,x^i$, this equation can be written in the form:

$$F^i = -\frac{\partial}{\partial x^i}\left(\frac{1}{2m}\frac{\partial S}{\partial x_j}\frac{\partial S}{\partial x^j} - \frac{\partial S}{\partial t}\right) \qquad [3\text{-}16d]$$

i.e.,

$$\mathbf{F} = -\,\nabla\left(-\frac{[\nabla S]^2}{2m} - \frac{\partial S}{\partial t}\right) \qquad [3\text{-}16e]$$

Employing equation [3-11]:

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[11] Belinfante F.J., *A Survey of Hidden Variable Theories,* p. 185. Pergamon, Oxford (1973).

$$\frac{[\nabla S]^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = -\frac{\partial S}{\partial t}$$

the force equation [3-16e] then becomes:

$$\mathbf{F} = -\boldsymbol{\nabla}\left[V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \right] \qquad [3\text{-}17]$$

Hence, referring to equations [3-12] and [3- 14], we can write:

$$F = -\boldsymbol{\nabla} PE \qquad [3\text{-}18a]$$

$$= -\boldsymbol{\nabla}(V + Q) \qquad [3\text{-}18b]$$

3.2 Velocity as a Function of Position - Bohm's Equation of Motion

According to the minimalist version of Bohm's model (which views [3-18b] as superfluous), a Bohmian particle traces out a smooth trajectory and its velocity evolves in a continuous manner as determined by $\boldsymbol{\nabla}S$. There is a clear contrast between this Bohmian mechanics and classical mechanics. In classical mechanics, the Newtonian equation of motion involves the second derivative of the particle's position coordinate with respect to time:

$$m\frac{d^2x}{dt^2} = -\nabla V \qquad [3\text{-}19]$$

whereas Bohm's equation of motion only involves the first derivative:

$$m\frac{dx}{dt} = \nabla S \qquad [3\text{-}20]$$

This has the following consequences. Solving the Newtonian equation in order to determine the particle's trajectory $\mathbf{x}(t)$ requires performing two integrals with respect to time, whereas solving Bohm's equation to obtain $\mathbf{x}(t)$ requires only one time integral. It follows that two unknown constants of integration arise in the Newtonian case, but only one in the Bohmian case. Physically, this means that, in attempting to determine a particle's trajectory uniquely in this way, we need to specify both the initial position and the initial velocity in the Newtonian case but only the initial position in the Bohmian case.

Contrasting the two different mechanics further, Bohm's model has been described as "Aristotelian"[12]. This refers to the ancient "common sense" viewpoint, attributed to Aristotle, that all objects will eventually come to rest unless kept moving by a force. In the subsequent physics of Newton, on the other hand, a moving object keeps on moving uniformly in a straight line unless acted upon by a net force. It has been argued by other authors (see footnote 10) that the above two equations of motion can be considered to exhibit this distinction in the following sense. Looking at the Newtonian equation, suppose the external influence is "switched off", which in this case means setting the potential V equal to zero. The particle's acceleration then becomes zero, but its velocity is not affected. If moving beforehand, the particle keeps moving in a uniform manner. In contrasting this result with Bohm's model, we will assume that [3-20] (in conjunction with the Schrodinger equation [3-2a]) is taken as providing a fundamental characterisation of Bohmian mechanics and that any other equations of the model are treated as secondary. We then suppose that the external influence can somehow be "switched off" in the Bohmian case, which this time means deleting the wavefunction accompanying the Bohmian particle so that R and S become zero (and the particle is left on its own). Setting S to zero in equation [3-20], we see that now it is the velocity that becomes zero and the particle jerks immediately to a halt.

[12] Durr D., Goldstein S. & Zanghi N., *Quantum Equilibrium and the Origin of Absolute Uncertainty,* Journal of Statistical Physics, Vol. 67, pp. 843-907 (1992).
Also Valentini A., *Pilot Wave Theory,* p. 47 in *Bohmian Mechanics and Quantum Theory: An Appraisal,* Edited by Cushing J.T., Fine A. and Goldstein S. Kluwer Academic Publishers, Dordrecht (1996).

3.3 Bohm's Model and Conventional Quantum Mechanics

Bohm has shown that, for a statistical ensemble of particles, the additional postulate $\mathbf{p} = \nabla S$, together with Born's statistical law $P(x) = R^2$, provides exact agreement with conventional non-relativistic quantum mechanics for all possible experimental circumstances[13]. This precise agreement means that Bohm's model cannot be experimentally distinguished from the conventional theory. Bohm's scheme is mathematically deterministic in the sense that the equation $\mathbf{p} = \nabla S$ uniquely determines a particle's future trajectory once the initial position is specified. However, as with classical mechanics, since it is not possible to measure or prepare the initial position with infinite precision, complete "predictability" cannot be achieved.

Bohm's model copes reasonably well with the Measurement Problem by postulating the existence of hidden variables which uniquely determine measurement outcomes (observations) as part of the measurement process. The variables (actually just the particle positions) are distributed such that the usual probabilities are obtained. Bohm's model also provides a comprehensible physical mechanism whereby the correct post-measurement statistical distributions for all quantum mechanical observables can be deduced[14] from the postulated pre-measurement position distribution $|\Psi(x)|^2$.

De Broglie[15] emphasized that the measurement process must allow us to distinguish between the different states u_n and that typically this means separating the different states (or something they interact with) in space. In simple cases, the outcome of the separation

[13] Bohm. D., *A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables*, Part 1: Physical Review Vol. 85, pp. 166-179 (1952).
[14] Bohm. D., *A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables*, Part 2: Physical Review Vol. 85, pp. 180- 193 (1952).
[15] de Broglie L., *Non-Linear Wave Mechanics.* Elsevier, Amsterdam (1960).

stage of the measurement is that the wavefunction $\Psi(\mathbf{x})$ evolves into a collection of spatially non-overlapping wave packets $c_1u_1(\mathbf{x}) + c_2u_2(\mathbf{x}) + \ldots$ As the packets gradually become spatially distinct, the particle (which is assumed to be travelling along a definite trajectory within the wavefunction) flows continuously and smoothly into one of them. The process of measurement is therefore completed simply by determining in which packet the particle is finally located.

The non-local aspects of Bohm's model are discussed in some detail in Appendix 1.

3.4 Energy and Momentum Not Conserved

Bohm's model proposes for quantum mechanics an underlying reality consisting of particles possessing continuous and smooth trajectories which are guided by a field whose properties are defined by the associated wavefunction Ψ. As shown earlier (see equations [3-11] to [3-13]), the Schrodinger equation can be manipulated to yield an equation containing terms that resemble a classical Hamiltonian:

$$\frac{[\nabla S]^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = -\frac{\partial S}{\partial t}$$

thereby pointing to the following expressions for potential energy PE and total energy E:

$$PE = V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

$$E = -\frac{\partial S}{\partial t} \quad (E = KE + PE)$$

Now, from [3-13] it follows that the total energy E of the particle is not constant, i.e., not conserved, except in the special case where the wavefunction's phase S depends linearly on the time t. (Similar considerations apply for momentum.) Classically, one would explain this lack of conservation by arguing that the particle is exchanging energy with the field with which it is interacting (i.e., the particle considered on its own is not a closed system). Here, the field is presumably the Schrodinger wave function. An “energy-momentum tensor” (which classically describes the energy and momentum content of a field) can be constructed for the Schrodinger field. Using this, however, the field’s total energy turns out to be separately conserved without involving the particle[16]. Hence the total energy of the particle-field system is not conserved either. This is in conflict with the situation everywhere else in physics.

A number of authors have suggested that this seems unsatisfactory[17] and that the absence of *"action and reaction"* between the guiding wave and the particle in Bohm’s theory represent a deficiency in the model. Holland[18], writing in *The Quantum Theory of Motion,* has summarised the situation as follows:

“One might expect the conservation laws would apply to the total field plus particle system in interaction, as in classical electrodynamics. The reason they do not is that the particle does not react back on the wave; the field satisfies its own conservation laws... From the standpoint of general theoretical principles this feature of the causal interpretation may appear as unsatisfactory, calling for a development of the theory to include a more symmetrical relation between wave and particle. At present we have no idea how a source term for the ψ-field could be consistently introduced into the dynamical equations in such a way that it does not disturb the empirically well-verified predictions of quantum theory...”

[16] Holland P.R., *The Quantum Theory of Motion,* Section 3.9.2, Cambridge University Press (1995).
[17] Cushing J.T., *Quantum Mechanics, Historical Contingency and the Copenhagen Hegemony,* p. 45. University of Chicago Press (1994).
[18] Holland P.R., *The Quantum Theory of Motion,* p. 120. Cambridge University Press (1995).

Anandan and Brown[19] have expressed similar reservations by asserting that the Bohm model fails to provide a satisfactory account of the nature of particle trajectories because its violation of the action-reaction principle prevents it being dynamically complete.

3.4.1 Restoring Conservation

The non-conservation of energy and momentum in Bohm's model can be traced to the fact that the model attempts to erect a particle interpretation using the standard Schrodinger equation, a field equation not containing any reference to the particle's position. This equation does not describe any influence of the particle on the field. Consequently, Bohm's quantum potential, which derives directly from the Schrodinger equation, appears to act unilaterally in the sense that the quantum potential acts on the particle (determining its trajectory) but the particle does not react back to change the magnitude of the field. This energy non-conservation deficiency in Bohm's model can be addressed by adding a source term to the Schrodinger equation which permits appropriate interaction between the particle and the field and in so doing reinstates the necessary conservation requirements. The problem with such a source term is, of course, that it is likely to interfere with the Schrodinger equation's highly successful agreement with experiment. In order for a model to be viable it is therefore necessary that the source term added be so constructed that the equation's empirically well-verified predictions remain intact. An aim of the subsequent chapters is to consider such a way in which the conservation principles can be incorporated within single-particle Bohmian mechanics.

[19] Anandan J. and Brown H.R., *On the Reality of Space-Time Geometry and the Wave-function,* Foundations of Physics, Vol. 25, pp. 349-360 (1995).

3.5 Extensions to Bohm's Model

Bohm's original model was constructed as a provisional point of view in an effort to provide new insight into quantum theory and suggest new possibilities for conceptual understanding. In particular, it aimed to show that the Copenhagen interpretation was not essential. A number of generalisations of Bohm's model have been shown to be possible. In each case, however, the assumption of a particle trajectory existing independently of measurement is central to the model. Consequently, Von Neumann's "Projection Postulate" is not required and the process of measurement can be understood satisfactorily. Bohm himself considered stochastic generalisations of his model[20], in which the quantity $\mathbf{v} = \nabla S/m$ becomes only the average velocity in a stochastic process and in which $P = R^2$ is the limiting distribution after allowing a sufficient period to establish a random diffusion. Subsequently, alternative generalisations have been developed as follows (these will be discussed further below):

- Holland exploited an additional angular degree of freedom that is already implicit in the Schrodinger equation.
- Deotto and Ghiradi added a term to the equation of continuity which maintains the required zero divergence. (Their models were not presented as serious proposals, but to make a point about nonuniqueness.)
- Sutherland relaxed the requirement $\mathbf{p} = \nabla S$ and considered a class of models

[20] Bohm D. and Hiley B.J., *Measurement Understood Through the Quantum Potential Approach,* Foundations of Physics, Vol. 14, pp. 255-274 (1984). See also Bohm D., *Proof that Probability Density Approaches* $|\psi|^2$ *in the Causal Interpretation of Quantum Theory*, Physical Review 89, pp. 458-466 (1953), and Bohm D. and Vigier J.P., *Model of the Causal Interpretation of Quantum Theory in Terms of a Fluid with Irregular Fluctuations*, Physical Review, Vol. 96, pp. 208-216 (1954).

described by joint probability distributions and satisfying the phase space continuity equation.

- Bohm's model has also been extended to include spin, perhaps most effectively by Bell[21].

3.5.1 Holland's Generalisation

In formulating extensions to Bohm's Model, Holland[22] has identified two important matters for consideration:

(i) Is the representation unique? Can we develop valid trajectory theories in representations other than the position representation described above? If so, how are the laws of motion in the various representations connected?

(ii) Within a specific representation, is the law of motion unique?

Holland's paper made the point that, in the absence of a canonical transformation theory[23] for the particle position and momentum variables in the de Broglie-Bohm theory, no general conclusions can be drawn as to connections between descriptions of motion in different representations. Beyond pointing this out and observing that the theory must be reconciled with results in position space, since all our measurements are finally made in the position representation, Holland's paper did not address point (i) in any significant way.

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[21] Bell J.S., *Speakable and Unspeakable in Quantum Mechanics, Paper 4: Introduction to the Hidden-Variable Question,* Cambridge University Press (1987).

[22] Holland P.R., *New Trajectory Interpretation of Quantum Mechanics,* Foundations of Physics, Vol. 28, pp. 881-911 (1998).

[23] A Canonical Transformation theory for particle position and momentum variables in the de Broglie-Bohm model would provide a standard form for expressing a change in the values for the position variables directly in terms of a change in the values for the momentum variables and vice versa.

With respect to the second point, Holland demonstrated that other deterministic trajectory interpretations can be produced by exploiting an internal angular degree of freedom in the Schrodinger equation. Holland's' argument develops from the observation that, without disturbing the density of the particles (given by $\rho = |\psi|^2$), a vector field having zero divergence ($\nabla . \mathbf{A} = 0$) can be added to the continuity equation in [3-3].

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \; ; \quad \mathbf{j} = \rho \, \mathbf{v}$$

(See also Deotto and Ghiradi below.) The addition of the divergenceless vector field permits the introduction of a variety of physically natural constraints to describe trajectories other than those specified by Bohm's equation of motion (**p** = **∇S)**. Holland argues that the Schrodinger equation tacitly involves a degree of freedom which is manifest when expressing the Schrodinger equation as a differential equation in an extended configuration space. Agreement with Bohm's model **p** = **∇S** is achieved when the new model is "averaged over the internal freedom." Under such circumstances, the predictions for Holland's formulation are indistinguishable from both Bohm's model and the standard Schrodinger formulation of quantum mechanics.

3.5.2 Deotto and Ghiradi's Generalisation

Deotto and Ghiradi[24] have presented a paper whose purpose was to investigate whether the Bohmian program of assuming that particles have definite trajectories leads unavoidably, when some general requirements of symmetry are taken into account, to Bohmian Mechanics. They concluded that there are infinitely many non-equivalent (from the point of view of trajectories) Bohmian models reproducing the predictions of

[24] Deotto E. and Ghirardi G.C., *Bohmian Mechanics Revisited,* Foundations of Physics Vol. 28, pp. 1-30 (1998).

quantum mechanics (because there are infinitely many terms with zero divergence that can be added to the Schrodinger current density).

3.5.3 Sutherland's Generalisation

Sutherland[25] has presented a non-relativistic, single-particle generalisation of Bohm's model based on the observation that the restriction $P(\mathbf{x}) = |\psi|^2$ is essential to Bohm's theory of measurement, whereas $\mathbf{p} = \nabla S$ is not. Sutherland's generalisation therefore relinquishes the momentum relationship and allows a spread of momentum values at each position. He points out that the equation of continuity, which ensures compatibility with continuous trajectories, remains valid provided the less restrictive relationship $\langle\mathbf{p}\rangle_{\mathbf{x}} = \nabla S(\mathbf{x})$ is satisfied, where the notation $\langle\mathbf{p}\rangle_{\mathbf{x}}$ stands for the mean value of momentum $\mathbf{p}$ at position $\mathbf{x}$. Having thus characterised a class of suitable models, Sutherland then constructs a particular generalisation of the de Broglie-Bohm model by choosing a specific joint distribution $P(\mathbf{x},\mathbf{p})$ for the particle's position and momentum. He then formulates an underlying dynamics for the motion of the particles such that the ensemble continues to conform to the chosen distribution through time. In his generalisation of the de Broglie-Bohm model, the expression obtained for $d\mathbf{p}/dt$ shows that, as in the original model, the particles can follow smooth trajectories (i.e., trajectories containing no discontinuous changes in velocity).

Sutherland's paper has some relevance to the present work, as follows: A Lagrangian density expression will be introduced here in a subsequent chapter with the aim of reinstating conservation of energy. In terms of the quantum potential Q (with the classical

[25] Sutherland R.I., *Phase Space Generalisation of the de Broglie-Bohm Model,* Foundations of Physics Vol. 27, pp. 845-863 (1997).

potential V ignored for simplicity), this Lagrangian expression leads to an equation of motion of the "Newtonian" form (as shown in equation [3-19]):

$$m \frac{d^2x}{dt^2} = -\nabla Q$$

rather than of the "Aristotelian" form [3-20]:

$$m \frac{dx}{dt} = \nabla S$$

This then means that the quantum potential [3-14]:

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

becomes relevant again, despite the arguments in the literature that this potential should be discarded from Bohm's model as superfluous. This apparent dilemma is, however, brought into better perspective by Sutherland's work, which essentially presents a whole class of models, all of which are in agreement with the predictions of conventional quantum mechanics. Bohm's model is then seen to be just one model in this class and, in fact, the only one involving an Aristotelian equation of motion. This therefore shows that the Aristotelian form is not an essential feature of a trajectory model for quantum mechanics and thereby makes the proposed reintroduction of the equation $m \frac{d^2x}{dt^2} = -\nabla Q$ quite reasonable.