Appendix 8: Modified Klein-Gordon Equation

(An appendix to Chapter 7, Section 7.4)

The extra term that is added to the Klein-Gordon equation by the interaction part of our relativistic Lagrangian density will be deduced here by inserting [7-15]:

$$\text{å}_{\text{interaction}} = -\,\{c\,\sqrt{(\partial_\mu S)\,(\partial^\mu S)} - mc^2\}\;\rho_0$$

into Lagrange's equation [7-14]:

$$\partial_\mu \frac{\partial\,\text{ă}}{\partial(\partial_\mu\phi^*)} - \frac{\partial\,\text{ă}}{\partial\phi^*} = 0 \qquad [A8\text{-}1]$$

Evaluating the first of the two terms in [A8-1], we obtain

$$\begin{aligned}
\partial_\mu \frac{\partial\,\text{å}_{\text{interaction}}}{\partial(\partial_\mu\phi^*)} &= \partial_\mu \frac{\partial\,[-\,\{c\,\sqrt{(\partial_\nu S)\,(\partial^\nu S)} - mc^2\}\;\rho_0\,]}{\partial(\partial_\mu\phi^*)} \\
&= -\,c\;\partial_\mu \frac{\partial\,\sqrt{(\partial_\nu S)\,(\partial^\nu S)}}{\partial(\partial_\mu\phi^*)}\;\rho_0 \\
&= -\,c\;\partial_\mu\;\{\frac{1}{2\,\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;\frac{\partial[(\partial_\nu S)(\partial^\nu S)]}{\partial(\partial_\mu\phi^*)}\;\rho_0\} \\
&= -\,c\;\partial_\mu\;\{\frac{1}{2\,\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;[(\partial_\nu S)\,\frac{\partial(\partial^\nu S)}{\partial(\partial_\mu\phi^*)} + (\partial^\nu S)\,\frac{\partial(\partial_\nu S)}{\partial(\partial_\mu\phi^*)}]\;\rho_0\} \\
&= -\,c\;\partial_\mu\;\{\frac{\partial_\nu S}{\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;\frac{\partial(\partial^\nu S)}{\partial(\partial_\mu\phi^*)}\;\rho_0\}
\end{aligned}$$

[A8-2]

Using the fact that $(\partial^\nu S)$ can be written in terms of the wavefunction and its complex conjugate as follows:

$$\partial_\mu S = -\,\frac{i\hbar}{2}\,\{\frac{\partial_\mu\phi}{\phi} - \frac{\partial_\mu\phi^*}{\phi^*}\} \qquad [A8\text{-}3]$$

equation [A8-2] becomes

$$\begin{aligned}
\partial_\mu \frac{\partial\,\text{å}_{\text{interaction}}}{\partial(\partial_\mu\phi^*)} &= -\,c\;\partial_\mu\;\{\frac{\partial_\nu S}{\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;[\frac{\partial}{\partial(\partial_\mu\phi^*)}\;(-\,\frac{i\hbar}{2}\{\frac{\partial^\nu\phi}{\phi} - \frac{\partial^\nu\phi^*}{\phi^*}\})]\;\rho_0\} \\
&= \frac{i\hbar c}{2}\;\partial_\mu\;\{\frac{\partial_\nu S}{\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;[\frac{\partial}{\partial(\partial_\mu\phi^*)}\;(-\,\frac{\partial^\nu\phi^*}{\phi^*})]\;\rho_0\} \\
&= -\,\frac{i\hbar c}{2}\;\partial_\mu\;\{\frac{\partial_\nu S}{\sqrt{(\partial_\lambda S)\,(\partial^\lambda S)}}\;\frac{1}{\phi^*}\;\frac{\partial(\partial^\nu\phi^*)}{\partial(\partial_\mu\phi^*)}\;\rho_0\}
\end{aligned}$$

Applying the identity:

$$\frac{\partial(\partial^{\mu}\phi^{*})}{\partial(\partial_{\nu}\phi^{*})} \equiv g^{\mu\nu}$$

we then have:

$$\begin{aligned}
\partial_{\mu}\frac{\partial\,\text{å}_{\text{interaction}}}{\partial(\partial_{\mu}\phi^{*})} &= -\frac{i\hbar c}{2}\,\partial_{\mu}\,\{\frac{\partial_{\nu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\frac{1}{\phi^{*}}\,g^{\mu\nu}\,\rho_0\} \\
&= -\frac{i\hbar c}{2}\,\partial_{\mu}\,\{\frac{\partial^{\mu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\frac{1}{\phi^{*}}\,\rho_0\} \\
&= \frac{i\hbar c}{2}\,\frac{\partial^{\mu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\frac{\partial_{\mu}\phi^{*}}{\phi^{*2}}\,\rho_0 - \frac{i\hbar c}{2\phi^{*}}\,\partial_{\mu}\,\{\frac{\partial^{\mu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\rho_0\}
\end{aligned}$$

[A8-4]

This is our result for the first term of [A8-1]. Turning to the second term, we have:

$$\begin{aligned}
\frac{\partial\,\text{å}_{\text{interaction}}}{\partial\phi^{*}} &= \frac{\partial\,[\,-\{c\,\sqrt{(\partial_{\mu}S)\,(\partial^{\mu}S)}\,-mc^{2}\}\,\rho_0\,]}{\partial\phi^{*}} \\
&= -\,c\,\frac{\partial\sqrt{(\partial_{\mu}S)\,(\partial^{\mu}S)}}{\partial\phi^{*}}\,\rho_0 \\
&= -\,c\,\frac{1}{2\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\frac{\partial[(\partial_{\mu}S)(\partial^{\mu}S)]}{\partial\phi^{*}}\,\rho_0 \\
&= -\frac{c}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,(\partial_{\mu}S)\,\frac{\partial(\partial^{\mu}S)}{\partial\phi^{*}}\,\rho_0
\end{aligned}$$

and using [A8-3] this becomes:

$$\begin{aligned}
\frac{\partial\,\text{å}_{\text{interaction}}}{\partial\phi^{*}} &= -\frac{c}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,(\partial_{\mu}S)\,\frac{\partial}{\partial\phi^{*}}\,[-\frac{i\hbar}{2}\{\frac{\partial^{\mu}\phi}{\phi}-\frac{\partial^{\mu}\phi^{*}}{\phi^{*}}\}]\,\rho_0 \\
&= -\frac{i\hbar c}{2\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,(\partial_{\mu}S)\,\frac{\partial}{\partial\phi^{*}}\,[\frac{\partial^{\mu}\phi^{*}}{\phi^{*}}]\,\rho_0 \\
&= \frac{i\hbar c}{2}\,\frac{\partial_{\mu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\frac{\partial^{\mu}\phi^{*}}{\phi^{*2}}\,\rho_0
\end{aligned}$$

[A8-5]

Finally, combining [A8-4] and [A8-5] together and cancelling two terms, we obtain the result:

$$\partial_{\mu}\frac{\partial\,\text{å}_{\text{interaction}}}{\partial(\partial_{\mu}\phi^{*})} - \frac{\partial\,\text{å}_{\text{interaction}}}{\partial\phi^{*}} = -\frac{i\hbar c}{2\phi^{*}}\,\partial_{\mu}\,\{\frac{\partial^{\mu}S}{\sqrt{(\partial_{\lambda}S)\,(\partial^{\lambda}S)}}\,\rho_0\}$$

This is the extra term to be added to the Klein-Gordon equation. (Note that, if desired, this expression can be written in terms of ϕ and ϕ^*, instead of S, by employing [A8-3].)

In analogy to the non-relativistic analysis in Appendix 2, it is easily shown that the modified Klein-Gordon equation still yields the same expression for the particle's velocity as does the standard Klein-Gordon equation. This means that all the formalism earlier in chapter 7 remains valid, despite the extra term derived above.