Appendix 4: Schrodinger Energy-Momentum Tensor

(An appendix to Chapter 6, Section 6.4)

To derive the various parts of the energy-momentum tensor corresponding to the free-field portion of our Lagrangian density [5-1] (i.e., corresponding to the standard Schrodinger equation), we will apply the formula [6-23]:

$$T^{\mu\nu}_{\text{field}} = [\ \partial^{\mu}\psi \frac{\partial}{\partial(\partial_{\nu}\psi)} + \partial^{\mu}\psi^{*} \frac{\partial}{\partial(\partial_{\nu}\psi^{*})} - g^{\mu\nu}\ ]\ \text{å}_{\text{field}}$$

to the following terms of [5-1]:

$$\text{å}_{\text{field}} = \frac{\hbar^{2}}{2m}(\partial_{k}\psi^{*})\ (\partial^{k}\psi) + \frac{i\hbar}{2}(\psi^{*}\ \partial_{t}\psi - \psi\ \partial_{t}\psi^{*})$$

Expressions for T^{ij}, T^{i0}, T^{0i} and T^{00} (i,j = 1,2,3) must be derived separately. The results are:

(i)

$$\begin{aligned}
T^{ij}_{\text{field}} &= [\ \partial^{i}\psi \frac{\partial}{\partial(\partial_{j}\psi)} + \partial^{i}\psi^{*} \frac{\partial}{\partial(\partial_{j}\psi^{*})} - g^{ij}\ ]\ \text{å}_{\text{field}} \\
&= (\partial^{i}\psi)\ \frac{\hbar^{2}}{2m}\ (\partial_{k}\psi^{*})\ g^{jk} + (\partial^{i}\psi^{*})\ \frac{\hbar^{2}}{2m}\ \delta^{j}_{k}\ (\partial^{k}\psi) \\
&\quad - g^{ij}\ [\frac{\hbar^{2}}{2m}(\partial_{k}\psi^{*})\ (\partial^{k}\psi) + \frac{i\hbar}{2}(\psi^{*}\ \partial_{t}\psi - \psi\ \partial_{t}\psi^{*})] \\
&= \frac{\hbar^{2}}{2m}\ [(\partial^{i}\psi)\ (\partial^{j}\psi^{*}) + (\partial^{i}\psi^{*})\ (\partial^{j}\psi) - g^{ij}\ [(\partial_{k}\psi^{*})\ (\partial^{k}\psi)] \\
&\quad - g^{ij}\ \frac{i\hbar}{2}(\psi^{*}\ \partial_{t}\psi - \psi\ \partial_{t}\psi^{*})]
\end{aligned}$$

(ii)

$$\begin{aligned}
T^{i0}_{\text{field}} &= [\ \partial^{i}\psi \frac{\partial}{\partial(\partial_{t}\psi)} + \partial^{i}\psi^{*} \frac{\partial}{\partial(\partial_{t}\psi^{*})} - g^{i0}\ ]\ \text{å}_{\text{field}} \\
&= (\partial^{i}\psi)\ \frac{i\hbar}{2}\ \psi^{*} - (\partial^{i}\psi^{*})\ \frac{i\hbar}{2}\ \psi - 0 \\
&= \frac{i\hbar}{2}\ (\psi^{*}\partial^{i}\psi - \psi\partial^{i}\psi^{*})
\end{aligned}$$

(continued)

(iii)

$$
\begin{aligned}
T^{0i}_{\text{field}} &= [\, \partial^t\psi \, \frac{\partial}{\partial(\partial_i\psi)} + \partial^t\psi^* \, \frac{\partial}{\partial(\partial_i\psi^*)} - g^{0i} \,] \;\mathring{a}_{\text{field}} \\
&= (\partial_t\psi)\, \frac{\hbar^2}{2m}\, (\partial_k\psi^*)\, g^{ik} + (\partial_t\psi^*)\, \frac{\hbar^2}{2m}\, \delta^i_k\, (\partial^k\psi) - 0 \\
&= \frac{\hbar^2}{2m}\, [\, (\partial_t\psi)\, (\partial^i\psi^*) + (\partial_t\psi^*)\, (\partial^i\psi) \,]
\end{aligned}
$$

(iv)

$$
\begin{aligned}
T^{00}_{\text{field}} &= [\, \partial^t\psi \, \frac{\partial}{\partial(\partial_t\psi)} + \partial^t\psi^* \, \frac{\partial}{\partial(\partial_t\psi^*)} - g^{00} \,] \;\mathring{a}_{\text{field}} \\
&= (\partial_t\psi)\, \frac{i\,\hbar}{2}\, \psi^* - (\partial_t\psi^*)\, \frac{i\,\hbar}{2}\, \psi \\
&\quad - g^{00}\, [\frac{\hbar^2}{2m}(\partial_k\psi^*)\, (\partial^k\psi) + \frac{i\,\hbar}{2}(\psi^*\, \partial_t\psi - \psi\, \partial_t\psi^*)] \\
&= -\, \frac{\hbar^2}{2m}\, (\partial_k\psi^*)\, (\partial^k\psi)
\end{aligned}
$$