Chapter 8: Non-Relativistic Limit

In the previous chapter, a relativistic version of Bohm's model incorporating energy and momentum conservation has been successfully formulated. The task now is to take the non-relativistic limit of that formalism. This will provide us with a mathematical description which incorporates conservation into Bohm's original model.

In the relativistic case, the symmetry between space and time made it sufficient to consider a single tensor expression $T^{\mu\nu}$ ($\mu,\nu = 0,1,2,3$). However, in dealing with the non-relativistic limit, we must obtain separate expressions for each of the tensor components T^{ij}, T^{i0}, T^{0i} and T^{00} ($i,j = 1,2,3$). Separate expressions must then be evaluated for the divergence of the T^{ij} and T^{i0} together in the first instance and T^{0i} and T^{00} together in the second instance. This lengthens the analysis somewhat.

In taking the non-relativistic approximation, it will also be found that some subtleties have to be taken into account. These will be illustrated by focussing our attention initially on the energy-momentum tensor of the particle.

8.1 Non-Relativistic Energy-Momentum Tensor for the Particle

8.1.1 Physical Interpretation of $T^{\mu\nu}{}_{particle}$

From equation [7-19], the relativistic expression for the particle's energy-momentum tensor is:

$$T^{\mu\nu}_{particle} = \rho_0 M u^\mu u^\nu \qquad [8\text{-}1]$$

In what follows, it needs to be kept in mind that the 4-velocity u^μ is defined to be $dx_0{}^\mu/d\tau$, where $x_0{}^\mu(\tau)$ is the particle's position at proper time τ. Note that u^μ is a function of τ whereas, in the non-relativistic limit, the 3-velocity $v^i = dx_0{}^i/dt$ is a function of the ordinary time t. Now, at first sight it would seem to be straightforward to take the non-relativistic limit of expression [8-1]. The rest density ρ_0 will become simply the density ρ:

$$\rho_0 \to \rho \qquad [8\text{-}2]$$

and de Broglie's variable mass M will reduce to the constant mass m:

$$M \to m \qquad [8\text{-}3]$$

Furthermore, since the proper time τ will become simply the ordinary time t:

$$\tau \to t \qquad [8\text{-}4]$$

the 4-velocity u^μ will reduce to 3-velocity for $\mu = 1,2,3$:

$$\frac{dx_0^i}{d\tau} \to \frac{dx_0^i}{dt} \qquad (i=1,2,3) \qquad [8\text{-}5]$$

and will reduce to the constant c for $\mu = 0$:

$$\frac{dx_0^0}{d\tau} \to c \qquad (\text{since } x^0 \equiv ct) \qquad [8\text{-}6]$$

Using the above limits then leads to the following result (with the expressions for T^{ij}, T^{i0}, T^{0i} and T^{00} written out separately):

$$T^{ij}{}_{particle} = \rho m v^i v^j \qquad (v^i \equiv \frac{dx_0^i}{dt}) \qquad [8\text{-}7a]$$

$$T^{i0}{}_{particle} = T^{0i} = \rho m v^i c \qquad [8\text{-}7b]$$

$$T^{00}{}_{particle} = \rho m c^2 \qquad [8\text{-}7c]$$

Examination of these expressions, however, raises two problems. First, factors of c are still present, even though the expressions are meant to be non-relativistic. Second, looking at the energy density term T^{00} in [8-7c], we see that the particle's rest energy mc^2 has been retained in taking the above limit, while its kinetic energy has been lost. This is not in keeping with the standard non-relativistic notion that the rest energy is a constant which plays no role and can be ignored (even though it is usually larger than the kinetic energy). To resolve these matters, a more careful analysis will now be given which focuses on the physical interpretation of the initial expression [8-1].

As stated in [6-3a] to [6-3c], the various terms in $T^{\mu\nu}{}_{\text{particle}}$ describe densities and currents of both momentum and energy. In particular, using the relationship:

$$\begin{aligned} \rho_0\, u^\nu &= \rho\, \frac{c}{u^0}\, u^\nu \qquad \text{(from [7 –12])} \\ &= \rho\, \frac{d\tau}{dt}\, \frac{dx_0^\nu}{d\tau} \\ &= \rho\, \frac{dx_0^\nu}{dt} \end{aligned} \tag{8-8}$$

plus the 4-momentum definition:

$$p^\mu \equiv Mu^\mu \tag{8-9}$$

the relativistic expression [8-1] can be written as:

$$T^{\mu\nu}_{\text{particle}} = p^\mu\, \rho\, \frac{dx_0^\nu}{dt} \tag{8-10}$$

Then, writing out the spatial and temporal components of this separately, we have:

$$T^{ij}_{\text{particle}} = p^i\, \rho\, v^j \tag{8-11a}$$

$$T^{i0}_{\text{particle}} = p^i\, \rho\, c \tag{8-11b}$$

$$T^{0i}_{\text{particle}} = p^0\, \rho\, v^i \tag{8-11c}$$

$$T^{00}_{\text{particle}} = p^0\, \rho\, c \tag{8-11d}$$

Now, ρ in these expressions indicates a density and ρv^i indicates a current. Hence, noting that p^i is the particle's relativistic 3-momentum and $p^0 c$ is its relativistic energy, we can identify $p^i\rho$ as the momentum density of the particle, $p^i\rho v^j$ as momentum current, etc., and obtain:

$T^{ij}{}_{particle} \equiv$ momentum current [8-12a]

$T^{i0}{}_{particle} \equiv$ momentum density $\times$ c [8-12b]

$T^{0i}{}_{particle} \equiv$ energy current $\div$ c [8-12c]

$T^{00}{}_{particle}$ = energy density [8-12d]

Note that, despite the fact that the terms T^{i0} and T^{0i} refer to two different physical quantities, viz. momentum density and energy current, the tensor is nevertheless symmetric: $T^{i0} = T^{0i}$. This is because, in the relativistic domain, momentum density and energy current are equal apart from a constant factor.

At this point we will consider the special case of a free particle with constant momentum and energy, so that the following divergence equation holds:

$$\partial_\nu \, T^{\mu\nu}_{particle} = 0 \quad [8\text{-}13]$$

Breaking this up into separate spatial and temporal terms, we have:

$$\partial_j \, T^{ij}_{particle} + \partial_0 \, T^{i0}_{particle} = 0 \quad [8\text{-}14a]$$

$$\partial_j \, T^{0j}_{particle} + \partial_0 \, T^{00}_{particle} = 0 \quad [8\text{-}14b]$$

Now, inserting expressions [8-12] into equations [8-14], we note that all factors of c cancel and we obtain:

∂_j(momentum current) + ∂_t(momentum density) = 0 [8-15a]

∂_j(energy current) + ∂_t(energy density) = 0 [8-15b]

These two equations can be recognized as equations of continuity describing momentum and energy conservation. Examining [8-12] and [8-15] highlights the basic physical meaning of the energy-momentum tensor and its divergence. This essence should be maintained in going to the non-relativistic limit. In other words, the **non**-relativistic formalism for a free particle:

$$\partial_j\, T^{ij}_{\text{particle}} + \partial_t\, T^{i0}_{\text{particle}} = 0 \qquad [8\text{-}16a]$$

$$\partial_j\, T^{0j}_{\text{particle}} + \partial_t\, T^{00}_{\text{particle}} = 0 \qquad [8\text{-}16b]$$

should still have the interpretation presented in equations [8-15]. This fact will be used to obtain the correct non-relativistic form for $T^{\mu\nu}{}_{\text{particle}}$.

Now, the non-relativistic momentum and energy of a particle are m**v** and E, respectively. (Here, E is the particle's **total** non-relativistic energy, i.e., the sum of the kinetic and potential energies[1]). Incorporating this extra detail into equations [8-15], we obtain:

$$\partial_j(mv^i\, \rho v^j) + \partial_t(mv^i\, \rho) = 0 \qquad [8\text{-}17a]$$

$$\partial_j(E\, \rho v^j) + \partial_t(E\, \rho) = 0 \qquad [8\text{-}17b]$$

Comparing [8-16] with [8-17] then yields the results:

$$T^{ij}{}_{\text{particle}} = mv^i\, \rho v^j \qquad [8\text{-}18a]$$

$$T^{i0}{}_{\text{particle}} = mv^i\, \rho \qquad [8\text{-}18b]$$

$$T^{0i}{}_{\text{particle}} = E\, \rho v^i \qquad [8\text{-}18c]$$

$$T^{00}{}_{\text{particle}} = E\, \rho \qquad [8\text{-}18d]$$

These expressions do not suffer from the two problems mentioned earlier, i.e., there are no longer any factors of c present[2] and the rest energy mc^2 has been eliminated in favour

[1] The reason for specifying the total energy here, rather than just the kinetic energy, will become clear in the next section.

of kinetic plus potential energy. On the other hand, the symmetry $T^{i0} = T^{0i}$ has been lost. It is important to note the source of this non-symmetry. Essentially it is due to the fact that neglecting rest energy ends the similarity between momentum density and energy current that had existed in the relativistic domain. Now that we have a physical explanation for the lack of symmetry, the objection raised towards the end in chapter 6 has lost its efficacy[3] and expressions [8-18] can be adopted as the appropriate non-relativistic form for $T^{\mu\nu}{}_{\text{particle}}$.

8.1.2 Rules for obtaining the Non-Relativistic Limit

A systematic procedure for obtaining [8-18] can be summarized by the three rules set out below. This will be helpful later in considering the cases of $T^{\mu\nu}{}_{\text{field}}$ and $T^{\mu\nu}{}_{\text{interaction}}$. Starting with the relativistic expressions for T^{ij}, T^{i0}, T^{0i} and T^{00}, the rules are as follows:

1. Remove terms containing mc^2 from T^{0i} and T^{00}. (To keep the overall divergence zero, it may also be necessary to remove any term whose divergence would previously have cancelled with that of a deleted mc^2 term.)
2. Divide T^{i0} by c and multiply T^{0i} by c (to remove redundant factors of c from these two expressions).
3. Take the non-relativistic limit $c \to \infty$.

---

[2] Note that the various factors of c appearing in the relativistic case are needed to ensure that all the components of $T^{\mu\nu}$ have the same dimensions (i.e., units of energy density) and thereby ensure that time and space remain on an equal footing. This symmetry between time and space components is not necessary in the non-relativistic case.

[3] It was also mentioned near the beginning of chapter 6 that symmetry of the energy-momentum tensor is required for conservation of angular momentum in the relativistic case. For the non-relativistic realm, it turns out that angular momentum conservation is related to the symmetry of a different tensor, namely the "mass-momentum" tensor. This alternative tensor continues to satisfy $T^{i0} = T^{0i}$, but its divergence describes conservation of **mass** and momentum (instead of energy and momentum). The non-relativistic mass-momentum tensor for a particle has the form: $T^{ij} = \rho m v^i v^j$, $T^{i0} = T^{0i} = \rho m v^i$, $T^{00} = \rho m$.

The motivation for Rule 1 has already been discussed. Rule 2 can be obtained by returning to [7-20]:

$$\partial_\nu T^{\mu\nu}_{\text{particle}} = \rho_0 \frac{dp^\mu}{d\tau}$$

and breaking this relativistic equation up into separate spatial and temporal terms:

$$\partial_j\, T^{ij}_{\text{particle}} + \partial_0\, T^{i0}_{\text{particle}} = \rho_0 \frac{dp^i}{d\tau} \qquad [8\text{-}19a]$$

$$\partial_j\, T^{0j}_{\text{particle}} + \partial_0\, T^{00}_{\text{particle}} = \frac{\rho_0}{c} \frac{dE}{d\tau} \qquad [8\text{-}19b]$$

Noting the factors of c contained in the derivatives $\partial_0 \equiv \partial/\partial(ct)$, these equations can then be written as

$$\partial_j\, T^{ij}_{\text{particle}} + \partial_t \left(\frac{1}{c}\, T^{i0}_{\text{particle}}\right) = \rho_0 \frac{dp^i}{d\tau} \qquad [8\text{-}20a]$$

$$\partial_j\, (c\, T^{0j}_{\text{particle}}) + \partial_t\, T^{00}_{\text{particle}} = \rho_0 \frac{dE}{d\tau} \qquad [8\text{-}20b]$$

Now, in contrast to this relativistic case, the non-relativistic version should be:

$$\partial_j\, T^{ij}_{\text{particle}} + \partial_t\, T^{i0}_{\text{particle}} = \rho \frac{dp^i}{dt} \qquad [8\text{-}21a]$$

$$\partial_j\, T^{0j}_{\text{particle}} + \partial_t\, T^{00}_{\text{particle}} = \rho \frac{dE}{dt} \qquad [8\text{-}21b]$$

Comparing equations [8-20] with [8-21] term by term, it is then seen that Rule 2 is necessary for the correct non-relativistic limit to be obtained.

The three rules above can be summarized in equation form as follows:

$$T^{ij}_{\text{non-rel}} = \lim_{c\to\infty} T^{ij}_{\text{rel}} \qquad [8\text{-}22a]$$

$$T^{i0}_{\text{non-rel}} = \lim_{c\to\infty} (T^{i0}_{\text{rel}} \div c) \qquad [8\text{-}22b]$$

$$T^{0i}_{\text{non-rel}} = \lim_{c\to\infty} \{\, [\, T^{i0}_{\text{rel}} - (mc^2 \text{ terms})\,] \times c \,\} \qquad [8\text{-}22c]$$

$$T^{00}_{\text{non-rel}} = \lim_{c\to\infty} [\, T^{00}_{\text{rel}} - (mc^2 \text{ terms})\,] \qquad [8\text{-}22d]$$

A more formal derivation of the non-relativistic form for $T^{\mu\nu}{}_{particle}$ is given in the next section using the rules stated above.

8.1.3 Derivation of $T^{\mu\nu}{}_{particle}$

The non-relativistic expressions [8-18] will be derived here from the corresponding relativistic expression [8-1]:

$$\begin{aligned} T^{\mu\nu}_{particle} &= \rho_0 M u^\mu u^\nu \\ &= \rho_0 M \frac{dx_0^\mu}{d\tau} \frac{dx_0^\nu}{d\tau} \end{aligned} \qquad [8\text{-}23]$$

by using the three rules formulated in the previous section. We begin by using equation [8-8]:

$$\rho_0 u^\mu = \rho \frac{dx_0^\mu}{dt}$$

to rewrite expression [8-23] as:

$$\begin{aligned} T^{\mu\nu}_{particle} &= \rho M \frac{dx_0^\mu}{dt} \frac{dx_0^\nu}{d\tau} \\ &= \rho M \frac{dx_0^\mu}{dt} \frac{dx_0^\nu}{dt} \frac{dt}{d\tau} \end{aligned} \qquad [8\text{-}24]$$

Now, in formulating the non-relativistic limit, the following binomial expansion will be useful[4]:

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots \end{aligned} \qquad [8\text{-}25]$$

Inserting this into [8-24] and presenting the expressions for T^{ij}, T^{i0}, T^{0i} and T^{00} separately, we have:

$$T^{ij}_{\text{particle}} = \rho\, M\, v^i\, v^j\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

$$T^{i0}_{\text{particle}} = \rho\, M\, v^i\, c\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

$$T^{0i}_{\text{particle}} = \rho\, M\, c\, v^i\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

$$T^{00}_{\text{particle}} = \rho\, M\, c^2\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

which will be written more conveniently here in the form:

$$T^{ij}_{\text{particle}} = \rho\, M\, v^i\, v^j\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,] \tag{8-26a}$$

$$T^{i0}_{\text{particle}} = \rho\, M\, v^i\, c\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,] \tag{8-26b}$$

$$T^{0i}_{\text{particle}} = \rho\, \frac{v^i}{c}\, [\, Mc^2 + \frac{1}{2}Mv^2 + ...\,] \tag{8-26c}$$

$$T^{00}_{\text{particle}} = \rho\, [\, Mc^2 + \frac{1}{2}Mv^2 + ...\,] \tag{8-26d}$$

To proceed further, we refer back to [7-7]:

$$M = m + \frac{Q}{c^2}$$

which allows us to rewrite our equations as:

$$T^{ij}_{\text{particle}} = \rho\, M\, v^i\, v^j\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

$$T^{i0}_{\text{particle}} = \rho\, M\, v^i\, c\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

$$T^{0i}_{\text{particle}} = \rho\, \frac{v^i}{c}\, [\, mc^2 + Q + \frac{1}{2}Mv^2 + ...\,]$$

$$T^{00}_{\text{particle}} = \rho\, [\, mc^2 + Q + \frac{1}{2}Mv^2 + ...\,]$$

Now, applying Rule 1, i.e., deleting terms containing the rest energy mc^2 from the energy current expression T^{0i} and the energy density expression T^{00}, we obtain:

$$T^{ij}_{\text{particle}} = \rho\, M\, v^i\, v^j\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$$

---

[4] See, e.g., pp. 67 and 85 in Rindler W., *Special Relativity*, 2nd Ed., Oliver and Boyd, Edinburgh (1969).

$T^{i0}_{particle} = \rho\, M\, v^i\, c\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$

$T^{0i}_{particle} = \rho\, \frac{v^i}{c}\, [\, Q + \frac{1}{2}Mv^2 + ...\,]$

$T^{00}_{particle} = \rho\, [\, Q + \frac{1}{2}Mv^2 + ...\,]$

(The deletion of mc^2 from $T^{0i}{}_{particle}$ can be shown to balance its deletion from $T^{00}{}_{particle}$, so that the overall divergence remains zero. This will be verified later when the total divergence is evaluated in detail.) Proceeding on, we divide T^{i0} by c and multiply T^{0i} by c in accordance with Rule 2, which yields:

$T^{ij}_{particle} = \rho\, M\, v^i\, v^j\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$

$T^{i0}_{particle} = \rho\, M\, v^i\, [\, 1 + \frac{1}{2}\frac{v^2}{c^2} + ...\,]$

$T^{0i}_{particle} = \rho\, v^i\, [\, Q + \frac{1}{2}Mv^2 + ...\,]$

$T^{00}_{particle} = \rho\, [\, Q + \frac{1}{2}Mv^2 + ...\,]$

Finally, we take the non-relativistic limit $c \rightarrow \infty$ in accordance with Rule 3. This also requires using the result [8-3] plus the following known limit[5]:

(de Broglie's relativistic Q) $\rightarrow$ (Bohm's non-relativistic Q) [8-27]

The expressions resulting from this step are:

$T^{ij}_{particle} = \rho m v^i v^j$ [8-28a]

$T^{i0}_{particle} = \rho m v^i$ [8-28b]

$T^{0i}_{particle} = \rho v^i\, [\frac{1}{2}mv^2 + Q]$ [8-28c]

$T^{00}_{particle} = \rho\, [\frac{1}{2}mv^2 + Q]$ [8-28d]

i.e.,

[5] See p. 121 in: L. de Broglie, *Nonlinear Wave Mechanics*, Elsevier, Amsterdam (1960).

$T^{ij}_{particle} = \rho m v^i v^j$

$T^{i0}_{particle} = \rho m v^i$

$T^{0i}_{particle} = \rho v^i E$

$T^{00}_{particle} = \rho E$

where $E \equiv ½mv^2 + Q$. These equations are then seen to be expressions [8-18] as required.

Note that the reason for the lack of symmetry of $T^{\mu\nu}{}_{particle}$ in the non-relativistic case can be seen clearly by looking at equations [8-26]. In applying our rules to the components T^{i0} and T^{0i}, we keep the first order term but drop the second order one in T^{i0}, whereas in contrast we keep the second order term but drop the first order one in T^{0i}. Not surprisingly, this reversal results in the two expressions becoming different.

We are now in a position to find the appropriate non-relativistic expressions for $T^{\mu\nu}{}_{field}$ and $T^{\mu\nu}{}_{interaction}$.

8.2 Non-Relativistic Energy-Momentum Tensor for the Field

A possible form for the Schrodinger $T^{\mu\nu}{}_{field}$ has already been derived in Appendix 4 using the standard formula [6-23]:

$$T^{\mu\nu}_{field} = [\ \partial^\mu\psi \frac{\partial}{\partial(\partial_\nu\psi)} + \partial^\mu\psi^* \frac{\partial}{\partial(\partial_\nu\psi^*)} - g^{\mu\nu}\ ]\ \text{å}_{\ field}$$

but it is necessary here to determine whether that result is in agreement with the non-relativistic limit of the Klein-Gordon expression [7-35]:

$$T^{\mu\nu}_{field} = \frac{\hbar^2}{2m} \{(\partial^\mu\phi)(\partial^\nu\phi^*) + (\partial^\mu\phi^*)(\partial^\nu\phi) - g^{\mu\nu} [(\partial_\lambda\phi^*)(\partial^\lambda\phi\ ) - (\frac{mc}{h})^2 \phi^*\phi]\} \qquad [8\text{-}29]$$

To take this limit for comparison, we need to know the relationship between the Klein-Gordon wavefunction ϕ and the Schrodinger wavefunction ψ. This relationship, which is a standard formula of quantum mechanics[6], is as follows:

$$\phi = \psi\, e^{-\frac{i}{\hbar}mcx^0} \qquad (x^0 = ct) \qquad [8\text{-}30]$$

Inserting [8-30] in [8-29] yields

$$\begin{aligned} T^{\mu\nu}_{\text{field}} = \frac{\hbar^2}{2m} \{ & (\partial^\mu[\psi e^{-\frac{i}{\hbar}mcx^0}])\, (\partial^\nu[\psi^* e^{\frac{i}{\hbar}mcx^0}]) + (\partial^\mu[\psi^* e^{\frac{i}{\hbar}mcx^0}])\, (\partial^\nu[\psi e^{-\frac{i}{\hbar}mcx^0}]) \\ & - g^{\mu\nu} (\partial_\lambda[\psi^* e^{\frac{i}{\hbar}mcx^0}])\, (\partial^\lambda[\psi e^{-\frac{i}{\hbar}mcx^0}]) + g^{\mu\nu} (\frac{mc}{\hbar})^2\, \psi^*\psi \} \end{aligned}$$

and employing the identities [5-19]:

$\partial x^\nu/\partial x_\mu \equiv g^{\mu\nu}$

$\partial x^\nu/\partial x^\mu \equiv \delta^\nu{}_\mu$

we then obtain:

$$\begin{aligned} T^{\mu\nu}_{\text{field}} = \frac{\hbar^2}{2m} \{ & (\partial^\mu\psi)\, (\partial^\nu\psi^*) + \psi\psi^*(-\frac{imc}{\hbar})\, g^{0\mu}\, (\frac{imc}{\hbar})\, g^{0\nu} \\ & + (\partial^\mu\psi)\, \psi^*\, (\frac{imc}{\hbar})\, g^{0\nu} + \psi\, (-\frac{imc}{\hbar})\, g^{0\mu}\, (\partial^\nu\psi^*) \\ & + (\partial^\mu\psi^*)\, (\partial^\nu\psi) + \psi^*\psi\, (\frac{imc}{\hbar})\, g^{0\mu}\, (-\frac{imc}{\hbar})\, g^{0\nu} \\ & + (\partial^\mu\psi^*)\, \psi\, (-\frac{imc}{\hbar})\, g^{0\nu} + \psi^*\, (\frac{imc}{\hbar})\, g^{0\mu}\, (\partial^\nu\psi) \\ & - g^{\mu\nu} [\, (\partial_\lambda\psi^*)\, (\partial^\lambda\psi) + \psi^*\psi\, (\frac{imc}{\hbar})\, \delta^0_\lambda\, (-\frac{imc}{\hbar})\, g^{0\lambda} \\ & \qquad + (\partial_\lambda\psi^*)\, \psi\, (-\frac{imc}{\hbar})\, g^{0\lambda} + \psi^*\, (\frac{imc}{\hbar})\, \delta^0_\lambda\, (\partial^\lambda\psi)\,] \\ & + g^{\mu\nu}\, (\frac{mc}{\hbar})^2\, \psi^*\psi \} \end{aligned}$$

$$\begin{aligned} = \frac{\hbar^2}{2m} \{ & (\partial^\mu\psi)\, (\partial^\nu\psi^*) + (\partial^\mu\psi^*)\, (\partial^\nu\psi) + 2\, (\frac{mc}{\hbar})^2\, g^{0\mu}\, g^{0\nu}\, \psi\psi^* \\ & + \frac{imc}{\hbar} [\, g^{0\mu}\, (\psi^*\, \partial^\nu\psi - \psi\, \partial^\nu\psi^*) + g^{0\nu}\, (\psi^*\, \partial^\mu\psi - \psi\, \partial^\mu\psi^*) \\ & - g^{\mu\nu} [\, (\partial_\lambda\psi^*)\, (\partial^\lambda\psi) + \frac{imc}{\hbar}\, (\psi^*\, \partial^0\psi - \psi\, \partial^0\psi^*)\,]\, \} \end{aligned}$$

---

[6] See, e.g., p. 7 in Greiner W., *Relativistic Quantum Mechanics – Wave Equations*, 2nd Ed., Springer, Berlin (1997).

$$= \frac{\hbar^2}{2m} [(\partial^\mu\psi) (\partial^\nu\psi^*) + (\partial^\mu\psi^*) (\partial^\nu\psi)] + mc^2 g^{0\mu} g^{0\nu} \psi\psi^*$$
$$+ \frac{i\hbar c}{2} [g^{0\mu} (\psi^* \partial^\nu\psi - \psi \partial^\nu\psi^*) + g^{0\nu} (\psi^* \partial^\mu\psi - \psi \partial^\mu\psi^*)]$$
$$- g^{\mu\nu} [\frac{\hbar^2}{2m} (\partial_\lambda\psi^*) (\partial^\lambda\psi) + \frac{i\hbar c}{2} (\psi^* \partial^0\psi - \psi \partial^0\psi^*)] \qquad [8\text{-}31]$$

To proceed towards the non-relativistic approximation, we now apply Rule 1 and delete the mc^2 term from this expression. In doing so, it is necessary to keep the overall divergence of the energy-momentum tensor zero. This means it is also necessary here to remove the term that would previously have cancelled with the deleted mc^2 term. It is not difficult to identify this term, as follows. The divergence (∂_ν) of the mc^2 term would have given a result of the form:

$$mc^2 g^{0\mu} \partial^0(\psi\psi^*)$$

i.e., a term containing the factor $g^{0\mu}$. This could cancel only with another term containing $g^{0\mu}$. Hence, looking at [8-31], we deduce that it is the term:

$$(i\hbar c/2)\, g^{0\mu} (\psi^* \partial^\nu\psi - \psi \partial^\nu\psi^*)$$

that should also be deleted. (This conclusion will be verified more rigorously later by calculating the divergence in full.) On the above basis, [8-31] reduces to:

$$T^{\mu\nu}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^\mu\psi) (\partial^\nu\psi^*) + (\partial^\mu\psi^*) (\partial^\nu\psi)] + \frac{i\hbar c}{2} g^{0\nu} (\psi^* \partial^\mu\psi - \psi \partial^\mu\psi^*)$$
$$- g^{\mu\nu} [\frac{\hbar^2}{2m} (\partial_\lambda\psi^*) (\partial^\lambda\psi) + \frac{i\hbar c}{2} (\psi^* \partial^0\psi - \psi \partial^0\psi^*)] \qquad [8\text{-}32]$$

Continuing on, the relativistic limit is obtained by taking the speed of light to be essentially infinite. In taking this limit, we will need to consider the T^{ij}, T^{i0}, T^{0i} and T^{00} cases separately.

8.2.1 Non-Relativistic T^{ij}_{field}

For this case, [8-32] yields

$$T^{ij}_{field} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial^j\psi^*) + (\partial^i\psi^*) (\partial^j\psi)] + 0$$
$$- g^{ij} [\frac{\hbar^2}{2m} \{ (\partial_k\psi^*) (\partial^k\psi) + (\partial_0\psi^*) (\partial^0\psi) \} + \frac{i\hbar c}{2} (\psi^* \partial^0\psi - \psi\, \partial^0\psi^*)]$$

(where k = 1,2,3). Switching from x^0 to ct, this becomes

$$T^{ij}_{field} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial^j\psi^*) + (\partial^i\psi^*) (\partial^j\psi)]$$
$$- g^{ij} [\frac{\hbar^2}{2m} \{ (\partial_k\psi^*) (\partial^k\psi) + \frac{1}{c^2} (\partial_t\psi^*) (\partial_t\psi) \} + \frac{i\hbar}{2} (\psi^* \partial_t\psi - \psi\, \partial_t\psi^*)]$$

and taking the limit $c \to \infty$, our Schrodinger expression for T^{ij} is found to be

$$T^{ij}_{field} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial^j\psi^*) + (\partial^i\psi^*) (\partial^j\psi)] - g^{ij} [\frac{\hbar^2}{2m} (\partial_k\psi^*) (\partial^k\psi) + \frac{i\hbar}{2} (\psi^* \partial_t\psi - \psi\, \partial_t\psi^*)] \qquad [8\text{-}33]$$

8.2.2 Non-Relativistic T^{i0}_{field}

Inserting $\mu = i$, $\nu = 0$ into [8-32], we obtain

$$T^{i0}_{field} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial^0\psi^*) + (\partial^i\psi^*) (\partial^0\psi)] + \frac{i\hbar c}{2} (\psi^* \partial^i\psi - \psi\, \partial^i\psi^*) - 0$$
$$= \frac{\hbar^2}{2mc} [(\partial^i\psi) (\partial_t\psi^*) + (\partial^i\psi^*) (\partial_t\psi)] + \frac{i\hbar c}{2} (\psi^* \partial^i\psi - \psi\, \partial^i\psi^*)$$

Dividing through by c in accordance with Rule 2, this becomes:

$$T^{i0}_{field} = \frac{\hbar^2}{2mc^2} [(\partial^i\psi) (\partial_t\psi^*) + (\partial^i\psi^*) (\partial_t\psi)] + \frac{i\hbar}{2} (\psi^* \partial^i\psi - \psi\, \partial^i\psi^*)$$

and taking the limit $c \to \infty$, the Schrodinger expression for T^{i0} is then found to be:

$$T^{i0}_{field} = \frac{i\hbar}{2} (\psi^* \partial^i\psi - \psi\, \partial^i\psi^*) \qquad [8\text{-}34]$$

8.2.3 Non-Relativistic T^{0i}_{field}

Inserting $\mu = 0$, $\nu = i$ into [8-32], we obtain

$$T^{0i}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^0\psi) (\partial^i\psi^*) + (\partial^0\psi^*) (\partial^i\psi)] + 0 - 0$$
$$= \frac{\hbar^2}{2mc} [(\partial_t\psi) (\partial^i\psi^*) + (\partial_t\psi^*) (\partial^i\psi)]$$

Multiplying through by c in accordance with Rule 2, then gives:

$$T^{0i}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial_t\psi^*) + (\partial^i\psi^*) (\partial_t\psi)] \qquad [8\text{-}35]$$

This is our Schrodinger expression for $T^{0i}{}_{\text{field}}$, since there are no factors of c remaining to require the limit $c \rightarrow \infty$ to be taken.

8.2.4 Non-Relativistic $T^{00}{}_{\text{field}}$

Inserting $\mu = 0$ and $\nu = 0$ into [8-32], we obtain:

$$T^{00}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^0\psi) (\partial^0\psi^*) + (\partial^0\psi^*) (\partial^0\psi)] + \frac{i\hbar c}{2} (\psi^* \partial^0\psi - \psi \partial^0\psi^*)$$
$$- \frac{\hbar^2}{2m} (\partial_\lambda\psi^*) (\partial^\lambda\psi) - \frac{i\hbar c}{2} (\psi^* \partial^0\psi - \psi \partial^0\psi^*)$$
$$= \frac{\hbar^2}{m} (\partial^0\psi) (\partial^0\psi^*) - \frac{\hbar^2}{2m} (\partial_\lambda\psi^*) (\partial^\lambda\psi)$$
$$= \frac{\hbar^2}{mc^2} (\partial_t\psi) (\partial_t\psi^*) - \frac{\hbar^2}{2m} \{ (\partial_k\psi^*) (\partial^k\psi) + \frac{1}{c^2} (\partial_t\psi) (\partial_t\psi^*) \}$$

and taking the limit $c \rightarrow \infty$, the Schrodinger expression for T^{00} is found to be:

$$T^{00}_{\text{field}} = - \frac{\hbar^2}{2m} (\partial_k\psi^*) (\partial^k\psi) \qquad [8\text{-}36]$$

8.2.5 Overall Non-Relativistic Result for $T^{\mu\nu}{}_{\text{field}}$

Gathering together expressions [8-33] to [8-36], our non-relativistic form for $T^{\mu\nu}{}_{\text{field}}$ is:

$$T^{ij}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial^j\psi^*) + (\partial^i\psi^*) (\partial^j\psi)] - g^{ij} [\frac{\hbar^2}{2m} (\partial_k\psi^*) (\partial^k\psi) + \frac{i\hbar}{2} (\psi^* \partial_t\psi - \psi \partial_t\psi^*)] \qquad [8\text{-}37a]$$

$$T^{i0}_{\text{field}} = \frac{i\hbar}{2} (\psi^* \partial^i\psi - \psi \partial^i\psi^*) \qquad [8\text{-}37b]$$

$$T^{0i}_{\text{field}} = \frac{\hbar^2}{2m} [(\partial^i\psi) (\partial_t\psi^*) + (\partial^i\psi^*) (\partial_t\psi)] \qquad [8\text{-}37c]$$

$$T^{00}_{\text{field}} = - \frac{\hbar^2}{2m} (\partial_k\psi^*) (\partial^k\psi) \qquad [8\text{-}37d]$$

Comparison with Appendix 4 then shows that the two different derivations have yielded the same result.

8.3 Non-Relativistic Energy-Momentum Tensor – Interaction Component

The non-relativistic form for $T^{\mu\nu}{}_{\text{interaction}}$ will now be derived. From equation [7-42], the relativistic expression is

$$T^{\mu\nu}_{\text{interaction}} = -\frac{c\,(\partial^{\mu}S)\,(\partial^{\nu}S)\,\rho_0}{\sqrt{(\partial_{\alpha}S)\,(\partial^{\alpha}S)}} \qquad [8\text{-}38]$$

This expression can be written in terms of the Klein-Gordon wavefunction ϕ and its complex conjugate ϕ^*, instead of in terms of the phase S, by using [7-38]:

$$\partial_{\mu}S = -\frac{i\hbar}{2}\,\{\frac{\partial_{\mu}\phi}{\phi} - \frac{\partial_{\mu}\phi^*}{\phi^*}\} \qquad [8\text{-}39]$$

It will be more convenient, however, to proceed by first re-expressing [8-39] in terms of the Schrodinger wavefunction ψ, using the relationship [8-30] that connects ϕ and ψ:

$$\phi = \psi\, e^{-\frac{i}{\hbar}mcx^0} \qquad (x^0 = ct)$$

Inserting this relationship into [8-39] yields

$$\partial_{\mu}S = -\frac{i\hbar}{2}\,\{\frac{\partial_{\mu}[\psi e^{-\frac{i}{\hbar}mcx^0}]}{\psi e^{-\frac{i}{\hbar}mcx^0}} - \frac{\partial_{\mu}[\psi^* e^{\frac{i}{\hbar}mcx^0}]}{\psi^* e^{\frac{i}{\hbar}mcx^0}}\}$$

and employing the identities [5-19], this becomes:

$$\partial_{\mu}S = -\frac{i\hbar}{2}\,\{\frac{\partial_{\mu}\psi}{\psi} - \frac{imc}{\hbar}\,\delta^0_{\mu} - \frac{\partial_{\mu}\psi^*}{\psi^*} - \frac{imc}{\hbar}\,\delta^0_{\mu}\}$$
$$= \frac{i\hbar}{2}\,(\frac{\partial_{\mu}\psi^*}{\psi^*} - \frac{\partial_{\mu}\psi}{\psi}) - mc\,\delta^0_{\mu}$$

This result will now be inserted into equation [8-38] so that we obtain $T^{\mu\nu}{}_{\text{interaction}}$ expressed directly in terms of ψ:

$$T^{\mu\nu}_{\text{interaction}} = -\, c \,\frac{[\,\frac{i\hbar}{2}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi}) - mc\, g^{0\mu}\,]\,[\frac{i\hbar}{2}\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}) - mc\, g^{0\nu}\,]\,\rho_0}{\sqrt{[\,\frac{i\hbar}{2}\,(\frac{\partial_\alpha\psi^*}{\psi^*} - \frac{\partial_\alpha\psi}{\psi}) - mc\,\delta^0_\alpha\,]\,[\,\frac{i\hbar}{2}\,(\frac{\partial^\alpha\psi^*}{\psi^*} - \frac{\partial^\alpha\psi}{\psi}) - mc\, g^{0\alpha}\,]}} \qquad [8\text{-}40]$$

We will now focus briefly on just the denominator of this expression, which can be rewritten as follows:

$$\sqrt{[\,\frac{i\hbar}{2}\,(\frac{\partial_\alpha\psi^*}{\psi^*} - \frac{\partial_\alpha\psi}{\psi}) - mc\,\delta^0_\alpha\,]\,[\,\frac{i\hbar}{2}\,(\frac{\partial^\alpha\psi^*}{\psi^*} - \frac{\partial^\alpha\psi}{\psi}) - mc\, g^{0\alpha}\,]}$$

$$= \sqrt{-\,\frac{\hbar^2}{4}\,(\frac{\partial_\alpha\psi^*}{\psi^*} - \frac{\partial_\alpha\psi}{\psi})\,(\frac{\partial^\alpha\psi^*}{\psi^*} - \frac{\partial^\alpha\psi}{\psi}) - i\hbar mc\,(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi}) + m^2c^2}$$

$$= mc\,\sqrt{-\,(\frac{\hbar}{2mc})^2\,(\frac{\partial_\alpha\psi^*}{\psi^*} - \frac{\partial_\alpha\psi}{\psi})\,(\frac{\partial^\alpha\psi^*}{\psi^*} - \frac{\partial^\alpha\psi}{\psi}) - \frac{i\hbar}{mc}\,(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi}) + 1}$$

Since the square root in this result will appear frequently in the rest of the present section, we will represent it using the letter K as follows:

$$K \equiv \frac{1}{\sqrt{-\,(\frac{\hbar}{2mc})^2\,(\frac{\partial_\alpha\psi^*}{\psi^*} - \frac{\partial_\alpha\psi}{\psi})\,(\frac{\partial^\alpha\psi^*}{\psi^*} - \frac{\partial^\alpha\psi}{\psi}) - \frac{i\hbar}{mc}\,(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi}) + 1}} \qquad [8\text{-}41]$$

Note that the non-relativistic limit of K is simply:

$$K \rightarrow 1 \qquad [8\text{-}42]$$

Returning to [8-40], the expression for $T^{\mu\nu}{}_{\text{interaction}}$ can now be written more simply as:

$$T^{\mu\nu}_{\text{interaction}} = -\,\frac{K}{m}\,[\,\frac{i\hbar}{2}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi}) - mc\, g^{0\mu}\,]\,[\frac{i\hbar}{2}\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}) - mc\, g^{0\nu}\,]\,\rho_0$$

$$= -\,\frac{K}{m}\,[\,-\,\frac{\hbar^2}{4}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi})\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}) - mc\,\frac{i\hbar}{2}\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi})\, g^{0\mu}$$
$$-\, mc\,\frac{i\hbar}{2}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi})\, g^{0\nu} + m^2c^2\, g^{0\mu}\, g^{0\nu}\,]\,\rho_0$$

$$= -\,K\,[\,-\,\frac{\hbar^2}{4m}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi})\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}) - \frac{i\hbar c}{2}\,(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi})\, g^{0\mu}$$
$$-\,\frac{i\hbar c}{2}\,(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi})\, g^{0\nu} + mc^2\, g^{0\mu}\, g^{0\nu}\,]\,\rho_0 \qquad [8\text{-}43]$$

We now apply Rule 1 and delete the mc^2 term from this expression. As in the case of $T^{\mu\nu}{}_{field}$ earlier, it is also necessary here to remove the term that would previously have cancelled with the deleted mc^2 term (in order to keep the overall divergence of the energy-momentum tensor zero). In order to identify this term, we note that the divergence (∂_ν) of the mc^2 term would have given a result of the form:

$$- mc^2 \, g^{0\mu} \, \partial^0(K\rho_0)$$

i.e., a term containing the factor $g^{0\mu}$. This can cancel only with another term containing $g^{0\mu}$. Hence, looking at [8-43], we must also delete the term:

$$- \frac{i\hbar c}{2} \left(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}\right) g^{0\mu}$$

(Again, this conclusion will be confirmed later when the full divergence is calculated.) With these two deletions, [8-43] reduces to:

$$T^{\mu\nu}_{interaction} = - K \left[- \frac{\hbar^2}{4m} \left(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi}\right) \left(\frac{\partial^\nu\psi^*}{\psi^*} - \frac{\partial^\nu\psi}{\psi}\right) - \frac{i\hbar c}{2} \left(\frac{\partial^\mu\psi^*}{\psi^*} - \frac{\partial^\mu\psi}{\psi}\right) g^{0\nu} \right] \rho_0 \qquad [8\text{-}44]$$

The relativistic approximation will now be obtained by taking the limit $c \to \infty$. In taking this limit, we will need to consider the T^{ij}, T^{i0}, T^{0i} and T^{00} cases separately.

8.3.1 Non-Relativistic $T^{ij}{}_{interaction}$

For this case, [8-44] yields:

$$T^{ij}_{interaction} = - K \left[- \frac{\hbar^2}{4m} \left(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}\right) \left(\frac{\partial^j\psi^*}{\psi^*} - \frac{\partial^j\psi}{\psi}\right) - 0 \right] \rho_0$$

and taking the non-relativistic limit via [8-2] and [8-42], we obtain:

$$T^{ij}_{interaction} = \frac{\hbar^2}{4m} \left(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}\right) \left(\frac{\partial^j\psi^*}{\psi^*} - \frac{\partial^j\psi}{\psi}\right) \rho \qquad [8\text{-}45]$$

8.3.2 Non-Relativistic $T^{i0}_{interaction}$

Inserting $\mu = i$, $\nu = 0$ into [8-44], we have:

$$T^{i0}_{interaction} = -K\,[\,-\frac{\hbar^2}{4m}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi}) - \frac{i\hbar c}{2}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho_0$$

$$= -K\,[\,-\frac{\hbar^2}{4mc}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi}) - \frac{i\hbar c}{2}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho_0$$

Dividing through by c in accordance with Rule 2, this becomes:

$$T^{i0}_{interaction} = -K\,[\,-\frac{\hbar^2}{4mc^2}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi}) - \frac{i\hbar}{2}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho_0$$

and taking the limit $c \to \infty$ then yields:

$$T^{i0}_{interaction} = \frac{i\hbar}{2}\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,\rho \qquad [8\text{-}46]$$

8.3.3 Non-Relativistic $T^{0i}_{interaction}$

Inserting $\mu = 0$, $\nu = i$ into [8-44], we obtain:

$$T^{0i}_{interaction} = -K\,[\,-\frac{\hbar^2}{4m}\,(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi})\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}) - 0\,]\,\rho_0$$

$$= -K\,[\,-\frac{\hbar^2}{4mc}\,(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho_0$$

Multiplying through by c in accordance with Rule 1 then gives:

$$T^{0i}_{interaction} = -K\,[\,-\frac{\hbar^2}{4m}\,(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho_0$$

and taking the non-relativistic limit via [8-2] and [8-42] then yields:

$$T^{0i}_{interaction} = \frac{\hbar^2}{4m}\,(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho \qquad [8\text{-}47]$$

8.3.4 Non-Relativistic $T^{00}_{interaction}$

Inserting $\mu = 0$ and $\nu = 0$ into [8-44], we obtain:

$$T^{00}_{\text{interaction}} = -K\,[\,-\frac{\hbar^2}{4m}(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi})(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi}) - \frac{i\hbar c}{2}(\frac{\partial^0\psi^*}{\psi^*} - \frac{\partial^0\psi}{\psi})\,]\,\rho_0$$

$$= -K\,[\,-\frac{\hbar^2}{4mc^2}(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi}) - \frac{i\hbar}{2}(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,]\,\rho_0$$

and taking the non-relativistic limit $c \to \infty$, together with [8-2] and [8-42], then yields:

$$T^{00}_{\text{interaction}} = \frac{i\hbar}{2}(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,]\,\rho \qquad [8\text{-}48]$$

8.3.5 Overall Non-Relativistic Result for $T^{\mu\nu}_{\text{interaction}}$

Gathering together expressions [8-45] to [8-48], our non-relativistic form for $T^{\mu\nu}_{\text{interaction}}$ is:

$$T^{ij}_{\text{interaction}} = \frac{\hbar^2}{4m}(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})(\frac{\partial^j\psi^*}{\psi^*} - \frac{\partial^j\psi}{\psi})\,\rho \qquad [8\text{-}49a]$$

$$T^{i0}_{\text{interaction}} = \frac{i\hbar}{2}(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,\rho \qquad [8\text{-}49b]$$

$$T^{0i}_{\text{interaction}} = \frac{\hbar^2}{4m}(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi})\,]\,\rho \qquad [8\text{-}49c]$$

$$T^{00}_{\text{interaction}} = \frac{i\hbar}{2}(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi})\,]\,\rho \qquad [8\text{-}49d]$$

8.4 Divergence and Conservation

The final task in this chapter is to check explicitly that the overall divergence of the non-relativistic energy-momentum tensor for the particle-field system is zero and thereby confirm that energy and momentum are conserved. Towards this end, the divergences of $T^{\mu\nu}_{\text{field}}$, $T^{\mu\nu}_{\text{particle}}$ and $T^{\mu\nu}_{\text{interaction}}$ will be evaluated separately.

8.4.1 Divergence of $T^{\mu\nu}{}_{field}$

There are two distinct parts to the divergence of the non-relativistic $T^{\mu\nu}{}_{field}$, corresponding to the cases μ = i (= 1,2,3) and μ = 0, respectively. For the first of these, we have (using expressions [8-37]):

$$\begin{aligned}
\partial_j T^{ij}_{field} + \partial_t T^{i0}_{field} = \partial_j \{ & \frac{\hbar^2}{2m} [(\partial^i\psi)(\partial^j\psi^*) + (\partial^i\psi^*)(\partial^j\psi)] \\
& - g^{ij} [\frac{\hbar^2}{2m} (\partial_k\psi^*)(\partial^k\psi) + \frac{i\hbar}{2} (\psi^* \partial_t\psi - \psi\, \partial_t\psi^*)] \} \\
& + \partial_t \{ \frac{i\hbar}{2} (\psi^* \partial^i\psi - \psi\, \partial^i\psi^*) \} \\
= & \frac{\hbar^2}{2m} [(\partial_j\partial^i\psi)(\partial^j\psi^*) + (\partial^i\psi)(\partial_j\partial^j\psi^*) + (\partial_j\partial^i\psi^*)(\partial^j\psi) + (\partial^i\psi^*)(\partial_j\partial^j\psi)] \\
& - \frac{\hbar^2}{2m} [(\partial^i\partial_k\psi^*)(\partial^k\psi) + (\partial_k\psi^*)(\partial^i\partial^k\psi)] \\
& - \frac{i\hbar}{2} [(\partial^i\psi^*)(\partial_t\psi) + \psi^* (\partial^i\partial_t\psi) - (\partial^i\psi)(\partial_t\psi^*) - \psi\, (\partial^i\partial_t\psi^*)] \\
& + \frac{i\hbar}{2} [(\partial_t\psi^*)(\partial^i\psi) + \psi^* (\partial_t\partial^i\psi) - (\partial_t\psi)(\partial^i\psi^*) - \psi\, (\partial_t\partial^i\psi^*)] \\
= & \frac{\hbar^2}{2m} [(\partial^i\psi)(\partial_j\partial^j\psi^*) + (\partial^i\psi^*)(\partial_j\partial^j\psi)] + \frac{i\hbar}{2} [(\partial^i\psi)(\partial_t\psi^*) - (\partial^i\psi^*)(\partial_t\psi)] \\
= & (\partial^i\psi^*) [\frac{\hbar^2}{2m} (\partial_j\partial^j\psi) - \frac{i\hbar}{2} (\partial_t\psi)] + (\partial^i\psi) [\frac{\hbar^2}{2m} (\partial_j\partial^j\psi^*) + \frac{i\hbar}{2} (\partial_t\psi^*)]
\end{aligned}$$

[8-50]

This can be simplified further by using the field equation corresponding to our Lagrangian density, i.e., by using the extended Schrodinger equation [5-21]:

$$\frac{\hbar^2}{2m}\partial_j\partial^j\psi - i\hbar\partial_t\psi = - \frac{i\hbar}{2\psi^*} \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \} \qquad [8\text{-}51]$$

Inserting [8-51] and its complex conjugate into [8-50], yields:

$$\begin{aligned}
\partial_j T^{ij}_{field} + \partial_t T^{i0}_{field} &= - \frac{i\hbar}{2\psi^*} (\partial^i\psi^*) \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \} + \frac{i\hbar}{2\psi} (\partial^i\psi) \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \} \\
&= \frac{i\hbar}{2} [\frac{1}{\psi} (\partial^i\psi) - \frac{1}{\psi^*} (\partial^i\psi^*)] \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}
\end{aligned}$$

and using the identity [5-14]:

$$\partial_j S = \frac{\hbar}{2i} [\frac{\partial_j\psi}{\psi} - \frac{\partial_j\psi^*}{\psi^*}]$$

we then obtain:

$$\partial_j T^{ij}_{field} + \partial_t T^{i0}_{field} = -(\partial^i S)\ \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}$$

$$= (\partial^i S)\ \{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} \qquad [8\text{-}52]$$

We now turn to the second part of the divergence of $T^{\mu\nu}{}_{field}$ (corresponding to $\mu = 0$). Using expressions [8-37], we have:

$$\partial_j T^{0j}_{field} + \partial_t T^{00}_{field} = \partial_j \{ \frac{\hbar^2}{2m} [(\partial^j\psi)(\partial_t\psi^*) + (\partial^j\psi^*)(\partial_t\psi)] \} + \partial_t [-\frac{\hbar^2}{2m} (\partial_k\psi^*)(\partial^k\psi)]$$

$$= \frac{\hbar^2}{2m} [(\partial_j\partial^j\psi)(\partial_t\psi^*) + (\partial^j\psi)(\partial_j\partial_t\psi^*) + (\partial_j\partial^j\psi^*)(\partial_t\psi) + (\partial^j\psi^*)(\partial_j\partial_t\psi)]$$
$$- \frac{\hbar^2}{2m} [(\partial_t\partial_k\psi^*)(\partial^k\psi) + (\partial_k\psi^*)(\partial_t\partial^k\psi)]$$

$$= \frac{\hbar^2}{2m} [(\partial_j\partial^j\psi)(\partial_t\psi^*) + (\partial_j\partial^j\psi^*)(\partial_t\psi)]$$

Using the field equation [8-51] and its complex conjugate, this can be re-expressed as:

$$\partial_j T^{0j}_{field} + \partial_t T^{00}_{field} = [i\hbar\partial_t\psi - \frac{i\hbar}{2\psi^*} \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}]\ \partial_t\psi^*$$

$$+ [-i\hbar\partial_t\psi^* + \frac{i\hbar}{2\psi} \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}]\ \partial_t\psi$$

$$= \frac{i\hbar}{2} [\frac{1}{\psi}(\partial_t\psi) - \frac{1}{\psi^*}(\partial_t\psi^*)]\ \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}$$

and, using the identity [5-14] again, we obtain:

$$\partial_j T^{0j}_{field} + \partial_t T^{00}_{field} = -(\partial_t S)\ \{ \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho \}$$

$$= (\partial_t S)\ \{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} \qquad [8\text{-}53]$$

8.4.2 Divergence of $T^{\mu\nu}{}_{particle}$

As with $T^{\mu\nu}{}_{field}$, there are two distinct parts to the divergence of the non-relativistic $T^{\mu\nu}{}_{particle}$, corresponding to the cases $\mu = 1,2,3$ and $\mu = 0$, respectively. In evaluating these two parts, it is necessary to keep in mind the following functional dependencies in the non-relativistic domain:

$$\mathbf{x_0} = \mathbf{x_0}(t) \neq \mathbf{x_0}(\mathbf{x}) \qquad [8\text{-}54a]$$

$$\mathbf{v} = \mathbf{v}(t) \neq \mathbf{v}(\mathbf{x}) \qquad [8\text{-}54b]$$

$$\rho = \rho[\mathbf{x} - \mathbf{x_0}(t)] = \rho[\mathbf{x_0}(t) - \mathbf{x}] \qquad [8\text{-}54c]$$

Using equations [8-18], the first part of the divergence is:

$$\begin{aligned}
\partial_j T^{ij}_{\text{particle}} + \partial_t T^{i0}_{\text{particle}} &= \partial_j\,(mv^i\,\rho v^j) + \partial_t\,(mv^i\,\rho) \\
&= mv^i v^j\,\partial_j\rho + m\,\partial_t(v^i\rho) \qquad \text{[since } v^i \neq v^i(\mathbf{x})\text{]} \\
&= mv^i v^j\,\frac{\partial\rho}{\partial x^j} + m\rho\,\partial_t v^i + mv^i\,\partial_t\rho \\
&= mv^i v^j\,(-\frac{\partial\rho}{\partial x_0^j}) + m\rho\,\frac{\partial v^i}{\partial t} + mv^i\,\frac{\partial\rho}{\partial x_0^j}\,\frac{\partial x_0^j}{\partial t} \\
&= -\,mv^i v^j\,\frac{\partial\rho}{\partial x_0^j} + m\rho\,\frac{dv^i}{dt} + mv^i\,\frac{\partial\rho}{\partial x_0^j}\,v^j \\
&= \rho\,m\,\frac{dv^i}{dt}
\end{aligned}$$

and referring back to [5-4], this can then be written as:

$$\partial_j T^{ij}_{\text{particle}} + \partial_t T^{i0}_{\text{particle}} = \rho\,\partial^i Q \qquad [8\text{-}55]$$

where Q is Bohm's non-relativistic quantum potential.

The second part of the divergence is:

$$\begin{aligned}
\partial_j T^{0j}_{\text{particle}} + \partial_t T^{00}_{\text{particle}} &= \partial_j\,(E\,\rho v^j) + \partial_t\,(E\,\rho) \\
&= \rho v^j\,\partial_j E + Ev^j\,\partial_j\rho + \rho\,\partial_t E + E\,\partial_t\rho \\
&= \rho v^j\,\partial_j E + Ev^j\,\frac{\partial\rho}{\partial x^j} + \rho\,\partial_t E + E\,\frac{\partial\rho}{\partial x_0^j}\,\frac{\partial x_0^j}{\partial t} \\
&= \rho v^j\,\partial_j E + Ev^j\,(-\frac{\partial\rho}{\partial x_0^j}) + \rho\,\partial_t E + E\,\frac{\partial\rho}{\partial x_0^j}\,\frac{dx_0^j}{dt}
\end{aligned}$$

$$= \rho v^j\,\partial_j E - E v^j\,\frac{\partial\rho}{\partial x_0^j} + \rho\,\partial_t E + E\,\frac{\partial\rho}{\partial x_0^j}\,v^j$$

$$= \rho v^j\,\partial_j E + \rho\,\partial_t E$$

$$= \rho v^j\,\partial_j(\tfrac{1}{2}mv^2 + Q) + \rho\,\partial_t(\tfrac{1}{2}mv^2 + Q)$$

$$= \rho v^j\,\partial_j Q + \rho\,\partial_t(-\tfrac{1}{2}mv_j v^j) + \rho\,\partial_t Q$$

$$= \rho v^j\,\partial_j Q + \rho\,\frac{dv^i}{dt}\,\frac{\partial}{\partial v^i}(-\tfrac{1}{2}mv_j v^j) + \rho\,\partial_t Q$$

$$= \rho v^j\,\partial_j Q - \rho m\,\frac{dv^i}{dt}\,\tfrac{1}{2}(g_{ij}v^j + v_j\delta_i^j) + \rho\,\partial_t Q$$

$$= \rho v^j\,\partial_j Q - \rho v_i\,m\,\frac{dv^i}{dt} + \rho\,\partial_t Q$$

Using [5-4], this can then be written as:

$$\partial_j T^{0j}_{\text{particle}} + \partial_t T^{00}_{\text{particle}} = \rho v^j\,\partial_j Q - \rho v_i\,\partial^i Q + \rho\,\partial_t Q$$

$$= \rho\,\partial_t Q \qquad [8\text{-}56]$$

8.4.3 Divergence of $T^{\mu\nu}{}_{\text{interaction}}$

From [8-49], our non-relativistic form for $T^{\mu\nu}{}_{\text{interaction}}$ is:

$$T^{ij}_{\text{interaction}} = \frac{\hbar^2}{4m}\,\left(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}\right)\left(\frac{\partial^j\psi^*}{\psi^*} - \frac{\partial^j\psi}{\psi}\right)\rho$$

$$T^{i0}_{\text{interaction}} = \frac{i\hbar}{2}\,\left(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}\right)\rho$$

$$T^{0i}_{\text{interaction}} = \frac{\hbar^2}{4m}\,\left(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi}\right)\left(\frac{\partial^i\psi^*}{\psi^*} - \frac{\partial^i\psi}{\psi}\right)]\,\rho$$

$$T^{00}_{\text{interaction}} = \frac{i\hbar}{2}\,\left(\frac{\partial_t\psi^*}{\psi^*} - \frac{\partial_t\psi}{\psi}\right)]\,\rho$$

and, using the identity [5-14], these expressions can be written more simply as:

$$T^{ij}_{\text{interaction}} = -\,\frac{(\partial^i S)\,(\partial^j S)}{m}\,\rho \qquad [8\text{-}57a]$$

$$T^{i0}_{\text{interaction}} = (\partial^i S)\,\rho \qquad [8\text{-}57b]$$

$$T^{0i}_{\text{interaction}} = -\frac{(\partial_t S)\,(\partial^i S)}{m}\,\rho \qquad [8\text{-}57c]$$

$$T^{00}_{\text{interaction}} = (\partial_t S)\,\rho \qquad [8\text{-}57d]$$

As with $T^{\mu\nu}{}_{\text{field}}$ and $T^{\mu\nu}{}_{\text{particle}}$, there are two distinct parts to the divergence of the non-relativistic $T^{\mu\nu}{}_{\text{interaction}}$. Employing expressions [8-57], the first part is:

$$\begin{aligned}
\partial_j T^{ij}_{\text{interaction}} + \partial_t T^{i0}_{\text{interaction}} &= \partial_j \left[-\frac{(\partial^i S)\,(\partial^j S)}{m}\,\rho \right] + \partial_t \left[(\partial^i S)\,\rho \right] \\
&= -(\partial^i S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \frac{(\partial^j S)\,\partial_j(\partial^i S)}{m}\,\rho + \rho\,\partial_t(\partial^i S) \\
&= -(\partial^i S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \frac{(\partial^j S)\,\partial^i(\partial_j S)}{m}\,\rho + \rho\,\partial^i(\partial_t S) \\
&= -(\partial^i S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \rho\,\partial^i \left[\frac{(\partial_j S)\,(\partial^j S)}{2m} - (\partial_t S) \right]
\end{aligned}$$

and, using [5-13], this can be written in terms of the quantum potential Q as:

$$\partial_j T^{ij}_{\text{interaction}} + \partial_t T^{i0}_{\text{interaction}} = -(\partial^i S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \rho\,\partial^i Q \qquad [8\text{-}58]$$

Similarly, the second part of the divergence is:

$$\begin{aligned}
\partial_j T^{0j}_{\text{interaction}} + \partial_t T^{00}_{\text{interaction}} &= \partial_j \left[-\frac{(\partial_t S)\,(\partial^j S)}{m}\,\rho \right] + \partial_t \left[(\partial_t S)\,\rho \right] \\
&= -(\partial_t S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \frac{(\partial^j S)\,\partial_j(\partial_t S)}{m}\,\rho + \rho\,\partial_t(\partial_t S) \\
&= -(\partial_t S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \frac{(\partial^j S)\,\partial_t(\partial_j S)}{m}\,\rho + \rho\,\partial_t(\partial_t S) \\
&= -(\partial_t S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \rho\,\partial_t \left[\frac{(\partial_j S)\,(\partial^j S)}{2m} - (\partial_t S) \right]
\end{aligned}$$

and using [5-13] again, this can be expressed more simply in terms of the potential Q, as follows:

$$\partial_j T^{0j}_{\text{interaction}} + \partial_t T^{00}_{\text{interaction}} = -(\partial_t S)\,\{ \partial_j(\rho\frac{\partial^j S}{m}) - \partial_t\rho \} - \rho\,\partial_t Q \qquad [8\text{-}59]$$

8.4.4 Divergence of $T^{\mu\nu}{}_{total}$

From equation [7-31] we have

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{field} + T^{\mu\nu}_{particle} + T^{\mu\nu}_{interaction}$$

The divergence of this overall energy-momentum tensor can now be obtained by combining the various results obtained above. As usual, the divergence will be written in two parts. First, using [8-52], [8-55] and [8-58], we have:

$$\partial_j T^{ij}_{total} + \partial_t T^{i0}_{total} = \left\{ \partial_j T^{ij}_{field} + \partial_t T^{i0}_{field} \right\} + \left\{ \partial_j T^{ij}_{particle} + \partial_t T^{i0}_{particle} \right\} + \left\{ \partial_j T^{ij}_{interaction} + \partial_t T^{i0}_{interaction} \right\}$$

$$= \left\{ (\partial^i S)\ \{\partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\} \right\} + \left\{ \rho\ \partial^i Q \right\} + \left\{ -\ (\partial^i S)\ \{\partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\} - \rho\ \partial^i Q \right\}$$

which cancels to:

$$\partial_j T^{ij}_{total} + \partial_t T^{i0}_{total} = 0 \qquad [8\text{-}60]$$

Second, using [8-53], [8-56] and [8-59], we have:

$$\partial_j T^{0j}_{total} + \partial_t T^{00}_{total} = \left\{ \partial_j T^{0j}_{field} + \partial_t T^{00}_{field} \right\} + \left\{ \partial_j T^{0j}_{particle} + \partial_t T^{00}_{particle} \right\} + \left\{ \partial_j T^{0j}_{interaction} + \partial_t T^{00}_{interaction} \right\}$$

$$= \left\{ (\partial_t S)\ \{\partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\} \right\} + \left\{ \rho\ \partial_t Q \right\} + \left\{ -\ (\partial_t S)\ \{\partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\} - \rho\ \partial_t Q \right\}$$

which cancels to:

$$\partial_j T^{0j}_{total} + \partial_t T^{00}_{total} = 0 \qquad [8\text{-}61]$$

Equations [8-60] and [8-61] are the desired results for energy and momentum conservation. (The divergence calculations above also serve as a useful double-check on our derivations of the non-relativistic expressions for $T^{\mu\nu}{}_{field}$, $T^{\mu\nu}{}_{particle}$ and $T^{\mu\nu}{}_{interaction}$.)

Therefore, from the viewpoint of conservation, a satisfactory non-relativistic model has been achieved.

8.5 Simplifications in the Bohmian Case

Some additional discussion is now needed to highlight the simplifications which occur in the above equations in the main case of interest. It will be helpful here to restate the three divergence results of Sec. 8.4 for further examination:

$$\partial_j\, T^{0j}_{field} + \partial_t\, T^{00}_{field} = (\partial_t S)\ \{\ \partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\ \} \qquad [8\text{-}53]$$

$$\partial_j T^{ij}_{particle} + \partial_t T^{i0}_{particle} = \rho\ \partial^i Q \qquad [8\text{-}55]$$

$$\partial_j T^{ij}_{interaction} + \partial_t T^{i0}_{interaction} = -\,(\partial^i S)\ \{\ \partial_j(\rho \frac{\partial^j S}{m}) - \partial_t \rho\ \} - \rho\ \partial^i Q \qquad [8\text{-}58]$$

In developing our Lagrangian formulation, it was necessary to suspend the Bohmian restriction **p** = **∇**S on the velocity of the particle. This meant we were actually considering a wide class of models, all of which satisfy the conservation laws for energy and momentum, but most of which need not be in agreement with the predictions of quantum theory. These models all satisfy the three divergence equations above. Note that, in general, the three different divergences (for $T^{\mu\nu}{}_{particle}$, $T^{\mu\nu}{}_{field}$ and $T^{\mu\nu}{}_{interaction}$) are all non-zero so that, for example, energy and momentum are being exchanged between the particle and the field.

Eventually, however, it is necessary to restore the restriction **p** = **∇**S in order to return to Bohm's model and agreement with experiment. This limits us to one particular model within the class considered. Since the whole class of models satisfies the energy and momentum conservation laws, the special model now singled out will do so as well. (The restriction is just an extra constraint which does not conflict with the earlier considerations in any way.) However, the assumption of no creation or annihilation of particles, in conjunction with **p** = **∇**S, simplifies the above divergence equations significantly, so that they become:

$$\partial_j\, T^{0j}_{\text{field}} + \partial_t\, T^{00}_{\text{field}} = 0 \qquad [8\text{-}62]$$

$$\partial_j T^{ij}_{\text{particle}} + \partial_t T^{i0}_{\text{particle}} = \rho\, \partial^i Q \qquad [8\text{-}63]$$

$$\partial_j T^{ij}_{\text{interaction}} + \partial_t T^{i0}_{\text{interaction}} = -\,\rho\, \partial^i Q \qquad [8\text{-}64]$$

In other words, in the special case of the Bohmian model singled out, the general relationship:

$$\partial_\nu\, T^{\mu\nu}_{\text{field}} + \partial_\nu\, T^{\mu\nu}_{\text{interaction}} + \partial_\nu\, T^{\mu\nu}_{\text{particle}} = 0 \qquad [8\text{-}65]$$

reduces to the two separate relationships:

$$\partial_\nu\, T^{\mu\nu}_{\text{field}} = 0 \qquad [8\text{-}66]$$

and

$$\partial_\nu\, T^{\mu\nu}_{\text{interaction}} + \partial_\nu\, T^{\mu\nu}_{\text{particle}} = 0 \qquad [8\text{-}67]$$

so that the formalism becomes somewhat less elegant in the Bohmian case[7]. This break-up into equations [8-66] and [8-67] is a necessary consequence of having a source-free wave equation. It tends, however, to disguise the fact that conservation is present, with the relationship [8-66] being particularly misleading in this regard. The apparent difficulty posed by this equation, as highlighted in the discussions of Sec. 6.4, has nevertheless been resolved by the necessary existence of the term $T^{\mu\nu}{}_{\text{interaction}}$, to which we have been led by examining Noether's theorem.

[7] An additional simplification is that the independent expressions for $T^{\mu\nu}{}_{\text{particle}}$ and $T^{\mu\nu}{}_{\text{interaction}}$ become connected by $T^{\mu\nu}_{\text{particle}} = -\,T^{\mu\nu}_{\text{interaction}}$

In conclusion, although the equations become rather simple in the one special case with which we are most concerned, this should not serve as a distraction from the successful reintroduction of energy and momentum conservation and the necessity of reaching it via the more general Lagrangian formulation employed.