Chapter 6: Energy-Momentum Tensors

This chapter outlines the general theory of energy and momentum conservation in terms of energy-momentum tensors, then applies these ideas to the case of Bohm's model. We will focus in particular on the case of a scalar field interacting with a particle. The rate of change of energy and momentum is described in terms of tensor divergence equations.

6.1 Basic Theory

With fields, we are concerned with densities, such as charge, probability, energy and momentum density. The treatment of the distribution of energy and momentum within the field proceeds in the same way as for the more familiar cases of charge and probability. Conservation of these latter quantities is described by a continuity equation involving both a density ρ and a current density $j^i = \rho v^i$ (i = 1,2,3):

$$\partial_i j^i + \partial_t \rho = 0 \qquad [6\text{-}1]$$

The densities that characterise a field's energy and momentum content are summarised in the form of the energy-momentum tensor $T^{\mu\nu}$. The various terms in this quantity correspond to energy and momentum densities and energy and momentum currents. In particular, the momentum density component in the i_{th} direction (for example, ρv^i in the case of a fluid having a mass density ρ and no internal stresses) will have a current component in the j_{th} direction ($\rho v^i v^j$ for a stressless fluid). Thus we are led to a description involving two indices i and j. In the relativistic case, the indices can separately have any of the values μ,ν = 0, 1, 2, or 3 and so the energy momentum tensor $T^{\mu\nu}$ consists of 16 components. In analogy to the continuity equation [6-1], energy and momentum conservation is described by the following set of 4 equations:

$$\partial_j T^{\mu j} + \partial_0 T^{\mu 0} = 0 \qquad [6\text{-}2]$$

The equation corresponding to $\mu = 0$ contains the terms T^{0j} and T^{00} and describes conservation of energy. The 3 equations corresponding to $\mu = i = 1, 2, 3$ contain the terms T^{ij} and T^{i0} and describe conservation of each component of momentum. In the relativistic case, it can be shown that conservation of angular momentum requires $T^{\mu\nu}$ to be symmetric in μ and ν and, as a consequence, the number of independent components is reduced from 16 to 10. These components have the following interpretation (ignoring any factors of c):

- T^{00} = energy density [6-3a]
- $T^{i0} = T^{0i}$ = three components of momentum density (equivalent to energy current) [6-3b]
- $T^{ij} = T^{ji}$ = six components of momentum current [6-3c]

Equations [6-2] can be written more elegantly as:

$$\partial_\nu T^{\mu\nu} = 0 \qquad [6\text{-}4]$$

6.2 Energy-Momentum Tensor for a Scalar Field

It can be shown[1] that the energy-momentum tensor for a real, free scalar field ϕ described by a Lagrangian density å is of the form:

$$T^{\mu\nu} = [\ \partial^\mu\phi \frac{\partial}{\partial(\partial_\nu\phi)} - g^{\mu\nu}\ ]\ \text{å} \qquad [6\text{-}5]$$

where we are using the notation

$$g^{\mu\nu} \equiv 1 \text{ for } \mu = \nu = 0 \qquad [6\text{-}6a]$$

$$\equiv -1 \text{ for } \mu = \nu = 1,2,3 \qquad [6\text{-}6b]$$

$$\equiv 0 \text{ for } \mu \neq \nu \qquad [6\text{-}6c]$$

$$\partial_\mu\phi \equiv \partial\phi/\partial x^\mu \qquad [6\text{-}6d]$$

$$\partial^\mu\phi \equiv g^{\mu\nu}\,\partial_\nu\phi \qquad [6\text{-}6e]$$

Using the Lagrangian density [4-10] for a real, massless scalar field:

$$\mathring{a}_{\text{scalar field}} = \tfrac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi)$$

the corresponding energy-momentum tensor is found from [6-5] to be:

$$T^{\mu\nu} = (\partial^\mu\phi)\,(\partial^\nu\phi) - \tfrac{1}{2}\,g^{\mu\nu}\,(\partial_\lambda\phi)\,(\partial^\lambda\phi) \qquad [6\text{-}7]$$

6.3 Energy and Momentum for a Scalar Field interacting with a Particle

From chapter 4, the overall Lagrangian density for describing a particle in interaction with a real, massless scalar field is given by [4-19]:

$$\mathring{a}_{\text{system}} = \tfrac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) + [\,\tfrac{1}{2}\,mv^2 - q\phi\,]\,\rho \qquad [6\text{-}8]$$

For a field that is exchanging energy and momentum with a particle, the basic condition expected for conservation is that the rate of change of the particle’s energy and momentum must exactly match the rate of change of the field’s energy and momentum, thus ensuring that the total remains constant. Also, the energy and momentum changes must also occur in a local manner. This means that the net energy and momentum flux into or out of the field in the immediate vicinity of the particle trajectory should match the particle's change of energy and momentum.

6.3.1 Energy and Momentum Conservation Equations

For the system characterised by the Lagrangian density [6-8] above, it is possible to derive conservation equations illustrating and confirming that changes in the field's momentum and energy are compensated by changes in the particle's momentum and energy. The 4-divergence of the field’s energy momentum tensor yields expressions corresponding to the rate of change of the field’s energy and momentum and consequently, as a first step towards obtaining the particle-field system conservation

[1] See, e.g., Ch. 12 in Goldstein H., *Classical Mechanics,* 2nd Ed. Addison-Wesley, Massachusetts (1980).

equations, we will evaluate the 4-divergence of the field's energy-momentum tensor [6-7]:

$$\partial_\nu T^{\mu\nu}_{\text{field}} = \partial_\nu [(\partial^\mu\phi) (\partial^\nu\phi) - ½ g^{\mu\nu} (\partial_\lambda\phi) (\partial^\lambda\phi)] \qquad [6\text{-}9a]$$

$$= (\partial_\nu\partial^\mu\phi) (\partial^\nu\phi) + (\partial^\mu\phi) (\partial_\nu\partial^\nu\phi) - \frac{1}{2} [(\partial^\mu\partial_\lambda\phi) (\partial^\lambda\phi) + (\partial_\lambda\phi) (\partial^\mu\partial^\lambda\phi)] \qquad [6\text{-}9b]$$

$$= (\partial_\lambda\partial^\mu\phi) (\partial^\lambda\phi) + (\partial^\mu\phi) (\partial_\nu\partial^\nu\phi) - \frac{1}{2} [(\partial^\mu\partial_\lambda\phi) (\partial^\lambda\phi) + (\partial^\mu\partial_\lambda\phi) (\partial^\lambda\phi)] \qquad [6\text{-}9c]$$

$$= (\partial^\mu\partial_\lambda\phi) (\partial^\lambda\phi) + (\partial^\mu\phi) (\partial_\nu\partial^\nu\phi) - (\partial^\mu\partial_\lambda\phi) (\partial^\lambda\phi) \qquad [6\text{-}9d]$$

$$= (\partial^\mu\phi) (\partial_\nu\partial^\nu\phi) \qquad [6\text{-}9e]$$

Now, the field equation which follows from the Lagrangian density [6-8] above is equation [4-20]:

$$\partial_\mu\partial^\mu\phi = - q \rho$$

which is simply the free-field equation with a source term added. Inserting this field equation into [6-9e] yields the tensor divergence equation:

$$\partial_\nu T^{\mu\nu}_{\text{field}} = - q \rho \partial^\mu\phi \qquad [6\text{-}10]$$

Returning again to the Lagrangian density [6-8], the particle equation of motion it yields via the integral equations [4-16] and Lagrange's equations [4-3] is the usual one for a particle in a scalar field:

$$\frac{dp^i}{dt} = q \partial^i\phi \qquad [6\text{-}11]$$

Also, from this equation for the rate of change of the particle's momentum, it is straightforward to derive an analogous one for the particle's energy (see Appendix 3):

$$\frac{dE}{dt} = q \frac{\partial\phi}{\partial t} \qquad [6\text{-}12]$$

We are now in a position to write down the equations we are seeking. Inserting [6-11] into the right hand side of [6-10], we obtain:

$$\partial_\nu T^{i\nu}_{\text{field}} = - \rho \frac{dp^i}{dt} \qquad (i = 1,2,3) \qquad [6\text{-}13]$$

Similarly, inserting [6-12] into [6-10], we obtain:

$$c\,\partial_\nu\, T^{0\nu}_{\text{field}} = -\,\rho\,\frac{dE}{dt} \qquad [6\text{-}14]$$

These are the two desired equations. They link the local changes in the field's momentum and energy to those of the particle, in accordance with the requirement of conservation.

Equations [6-13] and [6-14] also hold for other classical cases, such as a particle interacting with an electromagnetic field (see the Lagrangian density [4-21] earlier). In developing our Lagrangian approach to Bohm's model, it will be necessary for something similar to hold in the case of a Bohmian particle interacting with a Schrodinger field.

6.3.2 Introduction of $T^{\mu\nu}{}_{\text{particle}}$

In the case of a particle interacting with a scalar field, conservation of momentum and energy can also be expressed by introducing an energy-momentum tensor for the particle and writing the following set of divergence equations ($\mu,\nu = 0,1,2,3$):

$$\partial_\nu(T^{\mu\nu}{}_{\text{field}} + T^{\mu\nu}{}_{\text{particle}}) = 0 \qquad [6\text{-}15]$$

For a relativistic particle, $T^{\mu\nu}{}_{\text{particle}}$ has the form[2]:

$$T^{\mu\nu}{}_{\text{particle}} = \rho_0 m u^\mu u^\nu \qquad [6\text{-}16]$$

where ρ_0, m and u^μ are rest density, rest mass and 4-velocity, respectively. This expression for $T^{\mu\nu}{}_{\text{particle}}$ will be utilised in a later chapter. The set of equations [6-15] can be shown[3] to be equivalent to the relativistic versions of [6-13] and [6-14] provided expression [6-16] is chosen for $T^{\mu\nu}{}_{\text{particle}}$.

From [6-15], the conservation of the three components of momentum (i = 1,2,3) will be described by the equations

$$\partial_\nu(T^{i\nu}{}_{\text{field}} + T^{i\nu}{}_{\text{particle}}) = 0 \qquad [6\text{-}17a]$$

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[2] See, e.g., Sec. 10-1 in Adler R., Bazin M. and Schiffer M., *Introduction to General Relativity*, 2nd Ed. McGraw-Hill Kogakusha, Tokyo (1975).

and conservation of energy will be described by

$$\partial_\nu(T^{0\nu}{}_{field} + T^{0\nu}{}_{particle}) = 0 \quad [6\text{-}17b]$$

6.3.3 Global Equations

Equations [6-17] involve momentum and energy densities and ensure conservation "locally" at each point in space. On the other hand, the conservation of the **total** values of momentum and energy will be described by the following "global" equations (i = 1,2,3):

$$\frac{d}{dt}\,[\, p^i_{field} + p^i_{particle} \,] = 0 \quad [6\text{-}18a]$$

$$\frac{d}{dt}\,[\, E_{field} + E_{particle} \,] = 0 \quad [6\text{-}18b]$$

Equations [6-18a] and [6-18b] can be derived from the "local" versions [6-17a] and [6-17b] by integrating over all space:

$$\int_{-\infty}^{\infty} \partial_\nu T^{i\nu}_{field}\, d^3x + \int_{-\infty}^{\infty} \partial_\nu T^{i\nu}_{particle}\, d^3x = 0 \quad [6\text{-}19a]$$

$$\int_{-\infty}^{\infty} \partial_\nu T^{0\nu}_{field}\, d^3x + \int_{-\infty}^{\infty} \partial_\nu T^{0\nu}_{particle}\, d^3x = 0 \quad [6\text{-}19b]$$

The densities of momentum and energy will thereby be converted to total values. The equivalence of equations [6-19a] and [6-19b] to equations [6-18a] and [6-18b] will now be demonstrated. For both $T^{\mu\nu}{}_{field}$ and $T^{\mu\nu}{}_{particle}$, the required integral over space can be written out in detail as follows:

$$\int_{-\infty}^{\infty} \partial_\nu T^{\mu\nu}\, d^3x = \int_{-\infty}^{\infty} (\partial_0 T^{\mu 0} + \partial_1 T^{\mu 1} + \partial_2 T^{\mu 2} + \partial_3 T^{\mu 3})\, d^3x \quad [6\text{-}20]$$

Under the reasonable assumption that the energy-momentum tensor falls off to zero at plus and minus infinity (in any spatial direction), the last three terms of [6-20] will be zero and so only the term containing the time derivative survives:

$$\int_{-\infty}^{\infty} \partial_\nu T^{\mu\nu}\, d^3x = \partial_0 \int_{-\infty}^{\infty} T^{\mu 0}\, d^3x = \frac{1}{c}\frac{d}{dt}\int_{-\infty}^{\infty} T^{\mu 0}\, d^3x \quad [6\text{-}21]$$

[3] Felsager B., *Geometry, Particles and Fields*, Sec. 1-6, Springer-Verlag, NY (1998).

In the last equality, the partial derivative has been replaced by the total derivative because, after the spatial integration d^3x has been performed, only time dependence remains. With the aid of [6-21], equations [6-19a] and [6-19b] can be written as:

$$\frac{d}{dt}\int_{-\infty}^{\infty} T^{i0}_{\text{field}}\, d^3x + \frac{d}{dt}\int_{-\infty}^{\infty} T^{i0}_{\text{particle}}\, d^3x = 0 \qquad \text{[6-22a]}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} T^{00}_{\text{field}}\, d^3x + \frac{d}{dt}\int_{-\infty}^{\infty} T^{00}_{\text{particle}}\, d^3x = 0 \qquad \text{[6-22b]}$$

Now, T^{i0} and T^{00} can be identified from equations [6-3a] and [6-3b] earlier as momentum density and energy density, respectively. Therefore these equations reduce to the global equations [6-18a] and [6-18b] as required:

$$\frac{d}{dt}\,[\, p^i_{\text{field}} + p^i_{\text{particle}}\,] = 0$$

$$\frac{d}{dt}\,[\, E_{\text{field}} + E_{\text{particle}}\,] = 0$$

6.4 Tentative Application to Bohm's Model

Having summarised the relevant theoretical formalism, we will now attempt to employ it to introduce conservation of energy and momentum into Bohm's model. In doing so, it will be found that some difficulties arise. Fortunately these can all be overcome by a deeper and more careful analysis. We will look briefly here at the problems that are encountered as a pointer towards an appropriate course of action to follow in the next chapter.

As discussed in chapter 5, our proposed Lagrangian density for Bohm's model is:

$$\begin{aligned}
\mathcal{L} = {} & \frac{\hbar^2}{2m}[\partial_j\psi^*(\mathbf{x})]\,[\partial^j\psi(\mathbf{x})] + \frac{i\hbar}{2}[\psi^*(\mathbf{x})\,\partial_t\psi(\mathbf{x}) - \psi(\mathbf{x})\,\partial_t\psi^*(\mathbf{x})] && (\textit{field terms}) \\
& - \tfrac{1}{2}\, m\,\rho(\mathbf{x}-\mathbf{x_0})\, v_j\, v^j && (\textit{particle term}) \qquad \text{[5-1]} \\
& - \rho(\mathbf{x}-\mathbf{x_0})\, Q(\mathbf{x}) && (\textit{interaction term})
\end{aligned}$$

By analogy with the classical cases of a particle in a scalar or vector field, we tentatively expect that equation [6-15] will continue to hold:

$\partial_\nu(T^{\mu\nu}{}_{field} + T^{\mu\nu}{}_{particle}) = 0$

in the case of the system described by [5-1]. Equation [6-15] describes transfers of energy and momentum between the field and the particle, in accordance with the requirements of conservation. As a first step towards establishing whether this equation remains valid in our Bohmian case, we will derive the free-field energy-momentum tensor corresponding to [5-1].

Because our Lagrangian density is non-relativistic, it does not possess the sort of symmetry between space and time that is characteristic of relativistic Lagrangians. It is therefore necessary to obtain separate expressions for T^{ij}, T^{i0}, T^{0i} and T^{00} (i,j = 1,2,3), rather than just a single $T^{\mu\nu}$ expression (μ,ν = 0,1,2,3). This lengthens the derivation somewhat. The desired expressions are found from the free-field part of the Lagrangian density [5-1] by applying the formula:

$$T^{\mu\nu}_{field} = [\ \partial^\mu\phi \frac{\partial}{\partial(\partial_\nu\phi)} + \partial^\mu\phi^* \frac{\partial}{\partial(\partial_\nu\phi^*)} - g^{\mu\nu}\ ]\ \mathcal{L}_{field} \qquad [6\text{-}23]$$

which is a generalisation of equation [6-5] from the case of a real field to that of a complex field. The derivations are given in Appendix 4 and the results are:

$$\begin{aligned} T^{ij}_{field} = {} & \frac{\hbar^2}{2m}\ \{\ (\partial^i\psi)\ (\partial^j\psi^*) + (\partial^i\psi^*)\ (\partial^j\psi) - g^{ij}\ (\partial_k\psi^*)\ (\partial^k\psi)\ \} \\ & - g^{ij}\ \frac{i\,\hbar}{2}\ (\psi^*\ \partial_t\psi - \psi\ \partial_t\psi^*) \end{aligned} \qquad [6\text{-}24a]$$

$$T^{i0}_{field} = \frac{i\,\hbar}{2}\ \{\ \psi^*\ \partial^i\psi - \psi\ \partial^i\psi^*\} \qquad [6\text{-}24b]$$

$$T^{0i}_{field} = \frac{\hbar^2}{2m}\ \{\ (\partial_t\psi\ )\ (\ \partial^i\psi^*) + (\partial_t\psi^*)\ (\partial^i\psi)\ \} \qquad [6\text{-}24c]$$

$$T^{00}_{field} = -\ \frac{\hbar^2}{2m}\ (\partial_k\psi^*)\ (\partial^k\psi) \qquad [6\text{-}24d]$$

Looking at these expressions, our first difficulty is apparent. The energy-momentum tensor is not symmetric, since we have:

$$T^{i0} \neq T^{0i} \qquad [6\text{-}25]$$

whereas a symmetric tensor had been expected from the relativistic discussion earlier in this chapter. Techniques exist to symmetrise an energy-momentum tensor[4]. However, as later analysis will show, the present lack of symmetry should not simply be removed in this way. Instead, its significance should and will be examined carefully. This matter will be resolved in the next chapter.

Leaving this point and continuing on, we want to see whether the energy-momentum tensor above yields conservation by satisfying equations [6-13] and [6-14]. Of these two equations, it will be sufficient to discuss [6-13]:

$$\partial_\nu \, T^{i\nu}_{\text{field}} = -\,\rho \, \frac{dp^i}{dt} \qquad (i = 1,2,3)$$

For our present purpose, the divergence on the left of this equation needs to be split into separate space and time components, so that we have:

$$\partial_j \, T^{ij}_{\text{field}} + \partial_t \, T^{i0}_{\text{field}} = -\,\rho \, \frac{dp^i}{dt} \qquad [6\text{-}26]$$

To check whether the energy-momentum tensor summarized in equations [6-24] is consistent with this conservation condition, expressions [6-24a] and [6-24b] will be inserted into the left hand side of [6-26]. This is done in Appendix 5. For the usual non-relativistic situation of a single particle with no creation or annihilation, the following result is obtained:

$$\partial_j \, T^{ij}_{\text{field}} + \partial_t \, T^{i0}_{\text{field}} = 0 \qquad [6\text{-}27]$$

Hence, unlike the scalar and vector field cases discussed in the previous chapter, the divergence of the field's energy-momentum tensor is zero here even when there is field-particle interaction. This result is not consistent with equation [6-26] and forbids energy and momentum transfer between the field and the particle. Since we appear to need an equation like [6-26] to hold, we are faced with a second difficulty.

Of course, this zero divergence of the Schrodinger energy-momentum tensor is well known and is one reason why people have concluded that Bohm's model is not compatible with energy and momentum conservation[5]. Nevertheless, Noether's theorem assures us that the desired conservation must exist for the Lagrangian density we have chosen. A closer examination of Noether's theorem will be needed to resolve this problem. However, some insight into the course to be followed can be gained by considering another well-known case, viz., an electromagnetic field and a Dirac spinor field in interaction. Before the interaction between these two fields begins, the divergences of the tensors $T^{\mu\nu}{}_{\text{electromag}}$ and $T^{\mu\nu}{}_{\text{Dirac}}$ are, of course, separately zero:

$$\partial_\nu(T^{\mu\nu}{}_{\text{electromag}}) = 0 \qquad [6\text{-}28]$$

$$\partial_\nu(T^{\mu\nu}{}_{\text{Dirac}}) = 0 \qquad [6\text{-}29]$$

With the onset of the interaction, the expressions for $T^{\mu\nu}{}_{\text{electromag}}$ and $T^{\mu\nu}{}_{\text{Dirac}}$ do not change (i.e., they each still look the same), but their individual divergences are no longer zero[6]. Now, from our experience with the classical cases of a particle in a scalar or vector field, one might expect the following overall condition to hold:

$$\partial_\nu(T^{\mu\nu}{}_{\text{electromag}} + T^{\mu\nu}{}_{\text{Dirac}}) = 0$$

by analogy with [6-15]. However, this is not the case. The correct overall divergence equation contains an extra term, as follows[7]:

$$\partial_\nu(T^{\mu\nu}{}_{\text{electromag}} + T^{\mu\nu}{}_{\text{Dirac}} + T^{\mu\nu}{}_{\text{interaction}}) = 0 \qquad [6\text{-}30]$$

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[4] See, e.g., Ch. 3, Sec. 4 in Barut A., *Electrodynamics and Classical Theory of Fields and Particles.* Macmillan, N.Y. (1964).

[5] See p. 115 in Holland P.R., *The Quantum Theory of Motion.* Cambridge University Press (1995).

This example demonstrates that the appearance of an additional term $T^{\mu\nu}{}_{\text{interaction}}$ may sometimes be needed to achieve conservation. In so doing it suggests a way in which our second difficulty may be tackled.

Pursuing this possible approach, it seems at first sight that a suitable extra term $T^{\mu\nu}{}_{\text{interaction}}$ could be obtained simply by applying the square bracket in equation [6-23] to the interaction part of [5-1] to construct the tentative expression:

$$T^{\mu\nu}_{\text{interaction}} = [\ \partial^{\mu}\phi \frac{\partial}{\partial(\partial_{\nu}\phi)} + \partial^{\mu}\phi^{*} \frac{\partial}{\partial(\partial_{\nu}\phi^{*})} - g^{\mu\nu}\ ]\ \text{å}_{\text{interaction}} \qquad [6\text{-}31]$$

This does not lead to the correct result, however, as will be seen in the next chapter. Instead the problem will be resolved more systematically by showing from first principles the necessity of an extra term $T^{\mu\nu}{}_{\text{interaction}}$ and the precise form it must take.

[6] This change occurs due to the appearance of source terms in the two field equations used in evaluating the divergences.

[7] See Ch. 3, Sec. 4 in Rzewuski J., *Field Theory: Vol. 1, Classical Theory.* Iliffe Books, London (1967).