Chapter 5: A Lagrangian Formulation of Bohms Model

The Lagrangian formalism will now be used to construct a modified version of Bohm's model which addresses the non-conservation deficiencies characteristic of that model. We will limit ourselves to the single particle case. The first step is to propose an overall Lagrangian density for the particle-field system and show that it yields Bohm's equation of motion for the particle, plus a field equation consistent with the Schrodinger equation.

5.1 Proposed Lagrangian Density

The development of a Lagrangian density for describing the Bohmian system of a particle and Schrodinger field in interaction will proceed as outlined in the previous chapter. It will be assumed that the Lagrangian density consists of distinct "free-field", "particle" and "interaction" components. In line with the previous development, it is also assumed that:

(i) the terms of the free-field component are the familiar ones[1] for generating the Schrodinger equation,

(ii) the "particle" component has its standard form $\frac{1}{2}mv^2\rho$,

(iii) the "interaction" term is the usual one for a scalar field (see previous chapter).

Consequently, the proposed Lagrangian density for describing the Bohmian system of a particle and field in interaction is:

[1] See, e.g., p. 18 in Greiner W., *Relativistic Quantum Mechanics – Wave Equations,* 2nd Ed. Springer, Berlin (1994).

å =

$$\frac{\hbar^2}{2m}[\partial_j\psi^*(\boldsymbol{x})]\,[\partial^j\psi(\boldsymbol{x})] + \frac{i\hbar}{2}[\psi^*(\boldsymbol{x})\,\partial_t\psi(\boldsymbol{x}) - \psi(\boldsymbol{x})\,\partial_t\psi^*(\boldsymbol{x})] - V(\boldsymbol{x})\,\psi^*(\boldsymbol{x})\,\psi(\boldsymbol{x})$$

(*field terms*) [5-1]

$$-\tfrac{1}{2}\, m\, \rho(\boldsymbol{x} - \boldsymbol{x_0})\, v_j\, v^j$$ (*particle term*)

$$-\rho(\boldsymbol{x} - \boldsymbol{x_0})\,[V(\boldsymbol{x}) + Q(\boldsymbol{x})]$$ (*interaction term*)

Here:

- ψ is the wavefunction of the particle;
- ρ is the particle's "density distribution" through space (see [5-3] below);
- the quantities m, $\mathbf{x_0}(t)$ and v^i denote the particle's mass, position and velocity, respectively;
- $Q(\mathbf{x})$ is the quantum potential (which can be written out explicitly in terms of ψ);
- $V(\mathbf{x})$ is the classical electric potential applied externally;
- repeated indices denote summation as usual (with i,j = 1,2,3 and $v_i = -v^i$) and we are using the notation: $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x^i$;
- any charges associated with the two potentials have been absorbed into the symbols Q and V for simplicity.

In adopting this Lagrangian density we are treating the particle's velocity as an independent variable and temporarily suspending the usual de Broglie-Bohm restriction $\mathbf{p} = \nabla S$. This means that the particle is tentatively allowed a range of possible velocity values at each position **x**, with the only restriction on its velocity being the equation of motion $m\, d\mathbf{v}/dt = -\nabla Q$ that arises from our Lagrangian. However, once this formalism has yielded the required equations for energy and momentum conservation, the restriction $\mathbf{p} = \nabla S$ can then be imposed at the end (this being possible without introducing any inconsistency).

The quantity ρ has the form of a delta function representing the density distribution of the particle through space. It depends entirely upon the particle's trajectory:

$\mathbf{x_0}(t) = [x_0^1(t), x_0^2(t), x_0^3(t)]$ [5-2]

and expands as:

$\rho = \delta[\mathbf{x} - \mathbf{x_0}(t)]$ [5-3a]

$\quad = \delta[x^1 - x_0^1(t)]\, \delta[x^2 - x_0^2(t)]\, \delta[x^3 - x_0^3(t)]$ [5-3b]

The spatial dependence of the last two terms in the Lagrangian density should be carefully observed. After evaluating the full spatial integral ∫ å d^{3}x of the Lagrangian density to obtain the Lagrangian L, these two terms become the usual Lagrangian for describing a particle interacting with a scalar field. Note that the potentials in the interaction term become functions of $\mathbf{x_0}$ after the integration. In contrast, the field terms in the Lagrangian density are not functions of $\mathbf{x_0}$ (indeed they become spatially constant after the integration) and hence make no contribution when Lagrange's equations [4-3] are applied to derive the particle's equation of motion. Also note that the external potential V appears in both the field terms and the interaction term, contributing only to the field equation in the former case and only to the particle equation of motion in the latter. For simplicity in the subsequent development, V will no longer be included explicitly in the formulation.

It is useful to point out here for future reference that a more general formulation of our Lagrangian density is possible in which an additional, arbitrary constant k is included in both the particle and interaction terms of [5-1]. This modification would leave the resulting equation of motion for the particle unchanged and would simply introduce a factor k into the source term of the field equation, as will be discussed later in this chapter.

The Lagrangian density [5-1] is translationally and rotationally invariant. Since it is not an explicit function of the coordinates, one can conclude from Noether's Theorem that the system's energy and momentum will be conserved overall.

It will now be confirmed that variation of the particle's world line in [5-1] yields Bohm's equation of motion.

5.2 Derivation of Bohm's Equation of Motion from the Lagrangian Density

From Chapter 3, Bohm's equation of motion [3-18b] may be equivalently written as:

$$\frac{dp_i}{dt} = \frac{\partial Q}{\partial x^i} \qquad [5\text{-}4]$$

where Q is the quantum potential and where the external potential V has been neglected for simplicity. To obtain this equation from the proposed Lagrangian density [5-1], it is necessary to insert the Lagrangian:

$$L = \int_{-\infty}^{\infty} \mathcal{L} \; d^3x = (\text{spatially constant field terms}) - \frac{m}{2}\, v_j\, v^j - Q(\mathbf{x_0}) \qquad [5\text{-}5]$$

into Lagrange's equations [4-3]:

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x_0^i}$$

to obtain:

$$\frac{d}{dt}\frac{\partial}{\partial v^i}\left[-\frac{m}{2}\, v_j\, v^j - Q(\mathbf{x_0}) \right] = \frac{\partial}{\partial x_0^i}\left[-\frac{m}{2}\, v_j\, v^j - Q(\mathbf{x_0}) \right] \qquad [5\text{-}6]$$

This reduces to:

$$m\,\frac{dv_i}{dt} = \frac{\partial Q}{\partial x^i} \qquad [5\text{-}7]$$

which is the same as equation [5-4]. Hence it is apparent that the proposed Lagrangian density yields Bohm's equation of motion as required.

Note that this result would remain valid if, as proposed earlier, the Lagrangian density [5-1] were generalised by multiplying the particle and interaction terms by an arbitrary constant k.

5.3 Field Equation Deriving from the Proposed Lagrangian Density

The field equation arising from our Lagrangian density can be obtained by applying Lagrange’s equations for fields [4-7]. In performing this task, the interaction component of the Lagrangian Density in [5-1] is found to contribute a source term to the usual Schrodinger equation, as expected. Once this new term has been derived, its compatibility with the experimentally verified quantum mechanical predictions, as described by the Schrodinger equation, will be addressed.

The relevant form of Lagrange's equations is:

$$\partial_\mu \frac{\partial\, \breve{a}}{\partial\, (\partial_\mu \psi^*)} - \frac{\partial\, \breve{a}}{\partial \psi} = 0 \qquad [5\text{-}8]$$

Inserting the proposed Lagrangian density [5-1] yields (with V = 0):

$$\begin{aligned} &\partial_\mu \frac{\partial}{\partial(\partial_\mu \psi^*)} \left[\frac{\hbar^2}{2m} (\partial_j \psi^*)(\partial^j \psi) + \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{m}{2} \rho\, v_j v^j - \rho\, Q \right] \\ &- \frac{\partial}{\partial \psi^*} \left[\frac{\hbar^2}{2m} (\partial_j \psi^*)(\partial^j \psi) + \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{m}{2} \rho\, v_j v^j - \rho\, Q \right] = 0 \end{aligned} \qquad [5\text{-}9]$$

Now, the field terms in the Lagrangian density are the standard ones from which the free-field Schrodinger equation can be derived (see Greiner, footnote 1), so the corresponding terms in [5-9] simply become:

$$\begin{aligned}
&\partial_\mu \frac{\partial}{\partial(\partial_\mu\psi^*)} \left[\frac{\hbar^2}{2m} (\partial_j \psi^*) (\partial^j \psi) + \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) \right] \\
&- \frac{\partial}{\partial\psi^*} \left[\frac{\hbar^2}{2m} (\partial_j \psi^*) (\partial^j \psi) + \frac{i\hbar}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) \right] \\
&= \frac{\hbar^2}{2m} \partial_j\partial^j\psi - \frac{i\hbar}{2}\partial_t\psi - \frac{i\hbar}{2}\partial_t\psi \\
&= -\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi
\end{aligned} \qquad [5\text{-}10]$$

Furthermore, the particle term $\frac{m}{2}\rho\, v_j\, v^j$ of the Lagrangian density is not a function of Ψ^* and so the differentiations with respect to Ψ^* and $\partial_\mu\psi^*$ in [5-9] eliminate this term. The interaction term, however, contains the potential Q, which is a function of both Ψ^* and $\partial_\mu\psi^*$ when written out explicitly. It may therefore be concluded that [5-9] reduces to the following field equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi - \left[\partial_\mu \frac{\partial}{\partial(\partial_\mu\psi^*)} - \frac{\partial}{\partial\psi^*}\right] \rho\, Q = 0 \qquad [5\text{-}11]$$

In accordance with convention, this field equation will be written with the free-field terms on the left and the source term on the right:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi = \left[\partial_\mu \frac{\partial}{\partial(\partial_\mu\psi^*)} - \frac{\partial}{\partial\psi^*}\right] \rho\, Q \qquad [5\text{-}12]$$

The Lagrangian density [5-1] has thus yielded a modified Schrodinger equation. To obtain further insight into this equation, the source term needs to be written out in detail in terms of the wave function and its derivatives. In order to proceed towards this goal, it is necessary first to express the quantum potential Q as a function of ψ. As described earlier, this potential is usually written in the form [3-14]:

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

However, from section 3.1.3 (especially equation [3-16d]) it is clear that Q can also be expressed in the equivalent form:

$$Q = \frac{1}{2m}\, \partial_j S\, \partial^j S - \partial_t S \qquad [5\text{-}13]$$

This latter expression will actually be the more appropriate one for our present purpose, since the derivation of the former expression requires one to assume the standard Schrodinger equation, which we are in the process of modifying here. It is true that expression [5-13] is itself derived from the velocity given by the Schrodinger current density, but this is a somewhat weaker assumption. In any case, it will be shown in Appendix 2 that the usual Schrodinger velocity expression remains unmodified by the present considerations, thereby confirming the consistency of choosing expression [5-13].

Now, the potential Q stated in equation [5-13] may be written in a form more directly amenable to analysis in terms of the new field equation [5-12] by making use of the definition $\psi = Re^{iS/\hbar}$ and the consequent identity:

$$\partial_j S = \frac{\hbar}{2i} \left[\frac{\partial_j \psi}{\psi} - \frac{\partial_j \psi^*}{\psi^*}\right] \qquad [5\text{-}14]$$

We obtain for the potential Q the expression:

$$Q = -\frac{\hbar^2}{8m} \left[\frac{\partial_j \psi}{\psi} - \frac{\partial_j \psi^*}{\psi^*}\right] \left[\frac{\partial^j \psi}{\psi} - \frac{\partial^j \psi^*}{\psi^*}\right] + \frac{i\hbar}{2} \left[\frac{\partial_t \psi}{\psi} - \frac{\partial_t \psi^*}{\psi^*}\right] \qquad [5\text{-}15]$$

The form of the source term in the new field equation [5-12] may be explicitly evaluated by inserting [5-15] into [5-12] and evaluating the expression:

$$\left[\partial_\mu \frac{\partial}{\partial(\partial_\mu \psi^*)} - \frac{\partial}{\partial \psi^*}\right] \left\{ \rho \left(-\frac{\hbar^2}{8m} \left[\frac{\partial_j \psi}{\psi} - \frac{\partial_j \psi^*}{\psi^*}\right] \left[\frac{\partial^j \psi}{\psi} - \frac{\partial^j \psi^*}{\psi^*}\right] + \frac{i\hbar}{2} \left[\frac{\partial_t \psi}{\psi} - \frac{\partial_t \psi^*}{\psi^*}\right] \right) \right\} \qquad [5\text{-}16]$$

It will be convenient here to break the calculation into two parts, as follows:

Part A:

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu \psi^*)} \left\{ \rho \left(-\frac{\hbar^2}{8m} \left[\frac{\partial_j \psi}{\psi} - \frac{\partial_j \psi^*}{\psi^*}\right] \left[\frac{\partial^j \psi}{\psi} - \frac{\partial^j \psi^*}{\psi^*}\right] + \frac{i\hbar}{2} \left[\frac{\partial_t \psi}{\psi} - \frac{\partial_t \psi^*}{\psi^*}\right] \right) \right\} \qquad [5\text{-}17a]$$

Part B:

$$-\frac{\partial}{\partial\psi^*}\ \{\ \rho\ (-\frac{\hbar^2}{8m}\ [\frac{\partial_j\psi}{\psi}-\frac{\partial_j\psi^*}{\psi^*}]\ [\frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}]+\frac{i\hbar}{2}\ [\frac{\partial_t\psi}{\psi}-\frac{\partial_t\psi^*}{\psi^*}]\ )\ \} \qquad [5\text{-}17b]$$

Beginning with Part A, we have:

[5-17a] =

$$\partial_\mu\ \{-\frac{\hbar^2}{8m}\ (\ [\ 0-\frac{\partial}{\partial(\partial_\mu\psi^*)}\ (\frac{\partial_j\psi^*}{\psi^*})\ ]\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]+[\ \frac{\partial_j\psi}{\psi}-\frac{\partial_j\psi^*}{\psi^*}\ ]\ [\ 0-\frac{\partial}{\partial(\partial_\mu\psi^*)}\ (\frac{\partial^j\psi^*}{\psi^*})\ ]\ )\ \rho$$
$$+\frac{i\hbar}{2}\ [\ 0-\frac{\partial}{\partial(\partial_\mu\psi^*)}\ (\frac{\partial_t\psi^*}{\psi^*})\ ]\ \rho\ \}$$

$$=\partial_\mu\ \{-\frac{\hbar^2}{8m}\ (-[\ \frac{\delta^\mu_j}{\psi^*}\ ]\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]+[\ \frac{\partial_j\psi}{\psi}-\frac{\partial_j\psi^*}{\psi^*}\ ]\ [-\frac{g^{\mu j}}{\psi^*}\ ]\ )\ \rho+\frac{i\hbar}{2}\ [-\frac{\delta^\mu_t}{\psi^*}\ ]\ \rho\ \} \qquad [5\text{-}18]$$

where we have used the familiar identities:

$$\partial x^\nu/\partial x_\mu \equiv g^{\mu\nu} \qquad [5\text{-}19a]$$

$$\partial x_\nu/\partial x_\mu \equiv \delta^\mu{}_\nu \qquad [5\text{-}19b]$$

$$\partial t/\partial x_\mu \equiv \delta^\mu{}_t \qquad [5\text{-}19c]$$

Hence, continuing on by using the tensor rules:

$$\partial_\mu\ g^{\mu\nu} = \partial^\nu \qquad [5\text{-}19d]$$

$$\partial_\mu\ \delta^\mu{}_\nu = \partial_\nu \qquad [5\text{-}19e]$$

$$\partial_\mu\ \delta^\mu{}_t = \partial_t \qquad [5\text{-}19f]$$

we obtain:

$$[5\text{-}18] = \{-\frac{\hbar^2}{8m}\ \partial_j\ (-[\ \frac{1}{\psi^*}\ ]\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]-[\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]\ \frac{1}{\psi^*}\ )\ \rho-\frac{i\hbar}{2}\ \partial_t\ \frac{1}{\psi^*}\ \rho\ \}$$

$$=\frac{\hbar^2}{4m}\ \partial_j\ [\ \frac{1}{\psi^*}\ ]\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho-\frac{i\hbar}{2}\ (-\frac{\partial_t\psi^*}{\psi^{*2}}\ \rho+\frac{\partial_t\rho}{\psi^*})$$

$$=\frac{\hbar^2}{4m}\ (\ [-\frac{\partial_j\ \psi^*}{\psi^{*2}}\ ]\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho+[\ \frac{1}{\psi^*}\ ]\ \partial_j\ [\ \frac{\partial^j\psi}{\psi}-\frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho\ )+\frac{i\hbar}{2}\ (\ \frac{\partial_t\psi^*}{\psi^{*2}}\ \rho-\frac{\partial_t\rho}{\psi^*}) \qquad [5\text{-}20a]$$

This is our result for Part A. Turning to Part B, we have:

$$[5\text{-}17b] = \frac{\partial}{\partial\psi^*}\ \{\ \frac{\hbar^2}{8m}\ [\ \frac{\partial_j\psi}{\psi} - \frac{\partial_j\psi^*}{\psi^*}\ ]\ [\ \frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho - \frac{i\hbar}{2}\ [\ \frac{\partial_t\psi}{\psi} - \frac{\partial_t\psi^*}{\psi^*}\ ]\ \rho\ \}$$

$$= \frac{\hbar^2}{8m}\ (\ [\ 0 - \frac{\partial}{\partial\psi^*}\ (\frac{\partial_j\psi^*}{\psi^*})\ ]\ [\ \frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}\ ] + [\ \frac{\partial_j\psi}{\psi} - \frac{\partial_j\psi^*}{\psi^*}\ ]\ [\ 0 - \frac{\partial}{\partial\psi^*}\ (\frac{\partial^j\psi^*}{\psi^*})\ ]\ )\ \rho$$
$$- \frac{i\hbar}{2}\ [\ - \frac{\partial}{\partial\psi^*}\ (\frac{\partial_t\psi^*}{\psi^*})\ ]\ \rho$$

$$= \frac{\hbar^2}{8m}\ (\ [\ \frac{\partial_j\psi^*}{\psi^{*2}}\ ]\ [\ \frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}\ ] + [\ \frac{\partial_j\psi}{\psi} - \frac{\partial_j\psi^*}{\psi^*}\ ]\ [\ \frac{\partial^j\psi^*}{\psi^{*2}}\ ]\ )\ \rho - \frac{i\hbar}{2}\ [\ \frac{\partial_t\psi^*}{\psi^{*2}}\ ]\ \rho$$

$$= \frac{\hbar^2}{4m}\ [\ \frac{\partial_j\psi^*}{\psi^{*2}}\ ]\ [\ \frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho - \frac{i\hbar}{2}\ [\ \frac{\partial_t\psi^*}{\psi^{*2}}\ ]\ \rho \qquad [5\text{-}20b]$$

Now, adding together parts A and B (i.e., equations [5-20a] and [5-20b]) allows some cancellation, so that the following expression is found for the source term to go on the right hand side of the new field equation [5-12]:

$$[\partial_\mu \frac{\partial}{\partial(\partial_\mu\psi^*)} - \frac{\partial}{\partial\psi^*}]\ \rho\ Q = \frac{\hbar^2}{4m}\ (\ [\ \frac{1}{\psi^*}\ ]\ \partial_j\ [\ \frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}\ ]\ \rho\ ) - \frac{i\hbar}{2}\ \frac{\partial_t\rho}{\psi^*} \qquad [5\text{-}21]$$

Finally, using the identity [5-14], this source term can be written more simply as:

$$[\partial_\mu \frac{\partial}{\partial(\partial_\mu\psi^*)} - \frac{\partial}{\partial\psi^*}]\ \rho\ Q = \frac{i\hbar}{2\psi^*}\ [\ \partial_j\ (\rho\frac{\partial^j S}{m}) - \partial_t\ \rho\ ]$$
$$= -\ \frac{i\hbar}{2\psi^*}\ [\nabla\cdot(\rho\frac{\nabla S}{m}) + \partial_t\rho\ ] \qquad [5\text{-}22]$$

Summing up, the modified Schrodinger field equation that follows from the Lagrangian density [5-1] is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi = -\ \frac{i\hbar}{2\psi^*}\ \{\nabla\cdot(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} \qquad [5\text{-}23]$$

where $\rho(\mathbf{x}-\mathbf{x_0})$ is the delta function defined in [5-3] and $S(\mathbf{x})$ is the phase of the wave function as usual.

It may be helpful to finish by pointing out that this equation can, if desired, be written out fully in terms of ψ simply by replacing S with the expression:

$$S = \frac{\hbar}{2i} \ln\left(\frac{\psi}{\psi^*}\right) \qquad [5\text{-}24]$$

5.4 Consistency of the Derived Field Equation with Experiment

It will now be considered to what extent the new equation is compatible with the wealth of experimental evidence supporting the standard Schrodinger equation. As mentioned earlier in this chapter, an additional constant factor k could have been included in the particle and interaction terms of the Lagrangian density [5-1]. If this had been done, the constant k would then have appeared in the source term on the right of [5-23]. Since this constant is arbitrary, its value could be assumed very small and so the difference between the predictions of the standard and modified Schrodinger equations could then be asserted to be too tiny to detect experimentally. This would be a satisfactory way of reconciling the new source term with the known facts.

However, an alternative and more intriguing possibility also exists, as will now be discussed. The essential point to note is that the terms in the curly brackets on the right of [5-23] resemble those of a continuity equation [3-3]. In particular, once the restriction $\mathbf{v} = \nabla S/m$ is reimposed, this bracket equalling zero becomes the condition for conservation of the "matter" making up the particle (since ρ is the matter density). Since such conservation can be assumed in the non-relativistic domain, the new field equation [5-23] reduces to the standard Schrodinger equation [3-2a] within the latter's realm of applicability. In other words, the experimental predictions are unchanged as long as there is no particle creation or annihilation.

This is a surprising and thought-provoking result. It encourages us to pursue the present approach further. It also raises the question as to what extent, if any, new predictions are likely to arise in the relativistic domain. An experimentally distinguishable generalisation of quantum mechanics would be interesting. On the other hand, reinterpreting ρ as charge density, rather than matter density, may allow the relativistic predictions to remain unchanged as well. In any case, we know from Noether's theorem that energy and momentum conservation will be achieved either way.