Appendix 7: Relativistic Equation of Motion

(An appendix to Chapter 7, Sections 7.2 and 7.3)

A7.1 Derivation from the Relativistic Lagrangian Density

It will be shown here that the Lagrangian density [7-8] yields the correct equation of motion [7-5] for the particle. The action function corresponding to this $\mathscr{L}$ will have the form:

$$\begin{aligned} \text{action} &= \int \mathring{a}\ d^3x\, dt \\ &= \frac{1}{c}\int \mathring{a}\ d^4x \\ &= \frac{1}{c}\int (\mathring{a}_{\text{field}} + \mathring{a}_{\text{particle}} + \mathring{a}_{\text{interaction}})\ d^4x \end{aligned} \qquad \text{[A7-1]}$$

The required equation of motion can be obtained from first principles by varying the particle's world line in the action function. Since the term $\mathscr{L}_{\text{field}}$ in [A7-1] is not a function of the particle's world line, it can be ignored in the present considerations. The remaining part of the action function has the form:

$$\frac{1}{c}\int (\mathring{a}_{particle} + \mathring{a}_{interaction})\ d^4x \qquad \text{[A7-2]}$$

Rather than going right back to first principles, it is simpler to perform our derivation via a relativistic version of Lagrange's equation. The appropriate generalisation of the non-relativistic equation [4-3] is[1]:

$$\frac{d}{d\tau}\frac{\partial L}{\partial u^{\mu}} = \frac{\partial L}{\partial x_0^{\mu}} \qquad \text{[A7-3]}$$

This equation highlights a further consideration. We actually need a Lagrangian L to insert into this equation, not a Lagrangian density $\mathscr{L}$. Now, the partial action in [A7-2] is related to the required Lagrangian L via[2]:

$$\text{action} \mid \int L\, d\tau \qquad \text{[A7-4]}$$

[1] See, e.g., p. 329 in Goldstein H., *Classical Mechanics*, 2nd Ed. Addison-Wesley, Massachusetts (1980).

both the action and L being Lorentz scalar invariants. Using the 4-velocity definition:

$$u^0 = \frac{dx^0}{d\tau}$$

expression [A7-2] can also be written in the form:

$$\text{action} = \int \frac{u^0}{c} \left(\mathring{a}_{\text{particle}} + \mathring{a}_{\text{interaction}} \right) d^3x \, d\tau \qquad \text{[A7-5]}$$

Comparing [A7-4] and [A7-5], the desired Lagrangian is then seen to be:

$$L = \int_{-\infty}^{\infty} \frac{u^0}{c} \left(\mathring{a}_{\text{particle}} + \mathring{a}_{\text{interaction}} \right) d^3x$$

With the aid of [7-8], this expression can be written out in detail as:

$$L = \int_{-\infty}^{\infty} \frac{u^0}{c} \rho_0 \left[mc\sqrt{u_\mu u^\mu} + Q \frac{\sqrt{u_\mu u^\mu}}{c} \right] d^3x$$

which, using the definition [7-13] for ρ_0, becomes:

$$L = \int_{-\infty}^{\infty} \delta(\mathbf{x} - \mathbf{x}_0) \left[mc\sqrt{u_\mu u^\mu} + Q \frac{\sqrt{u_\mu u^\mu}}{c} \right] d^3x$$

$$= mc\sqrt{u_\mu u^\mu} + Q \frac{\sqrt{u_\mu u^\mu}}{c}$$

The particle's equation of motion is now found by inserting this Lagrangian into Lagrange's equation [A7-3], which yields the following result to be simplified:

$$\frac{d}{d\tau} \frac{\partial}{\partial u^\mu} \left[\left(mc + \frac{Q}{c}\right) \sqrt{u_\nu u^\nu} \right] = \frac{\partial}{\partial x_0^\mu} \left[\left(mc + \frac{Q}{c}\right) \sqrt{u_\nu u^\nu} \right]$$

Taking the u and x_0 derivatives we have:

$$\frac{d}{d\tau}\left[\left(mc + \frac{Q}{c}\right) \tfrac{1}{2} \left(u_\nu u^\nu\right)^{-1/2} 2u_\mu \right] = \sqrt{u_\nu u^\nu} \, \frac{\partial}{\partial x_0^\mu}\left(\frac{Q}{c}\right)$$

and using the identity $u_\mu u^\mu = c^2$ this reduces to

$$\frac{d}{d\tau}\left[\left(m + \frac{Q}{c^2}\right) u_\mu \right] = \frac{\partial Q}{\partial x_0^\mu} \qquad \text{[A7-6]}$$

Finally, employing the definition [7-7] for the variable rest mass M:

$$M = m + \frac{Q}{c^2}$$

[2] See, e.g., p. 203 in Anderson J.L., *Principles of Relativity Physics*, Academic Press, N.Y. (1967).

we obtain the equation of motion:

$$\frac{d(Mu_\mu)}{d\tau} = \frac{\partial Q}{\partial x_0^\mu}$$

or, equivalently:

$$\frac{dp_\mu}{d\tau} = \partial_\mu Q$$

in agreement with the expected result [7-5].

A7.2 Consistency of the Equation of Motion with the Identity $u_\mu u^\mu = c^2$

In deriving the equation of motion for the particle, the restriction $u_\mu u^\mu = c^2$ is temporarily suspended until after the variation process has been performed[3]. We will now carry out a standard check that the resultant equation of motion is then consistent with the identity $u_\mu u^\mu = c^2$ without any unwanted restrictions arising. For this purpose, it is most convenient to use the form shown in equation [A7-6]. Introducing u^μ on both sides of [A7-6], we obtain

$$u^\mu \frac{d}{d\tau}[\,(m + \frac{Q}{c^2})\, u_\mu\,] = u^\mu \frac{\partial Q}{\partial x_0^\mu}$$

which can be written as:

$$u^\mu u_\mu \frac{d}{d\tau}(m + \frac{Q}{c^2}) + (m + \frac{Q}{c^2})\, u^\mu \frac{du_\mu}{d\tau} = \frac{dQ}{d\tau}$$

i.e.,

$$u^\mu u_\mu \frac{d}{d\tau}(\frac{Q}{c^2}) + (m + \frac{Q}{c^2})\, ½\, \frac{d(u^\mu u_\mu)}{d\tau} = \frac{dQ}{d\tau}$$

Using the identity $u_\mu u^\mu = c^2$ we then have:

$$c^2 \frac{d}{d\tau}(\frac{Q}{c^2}) + (m + \frac{Q}{c^2})\, ½\, \frac{d(c^2)}{d\tau} = \frac{dQ}{d\tau}$$

i.e.,

$$\frac{dQ}{d\tau} + 0 = \frac{dQ}{d\tau} \qquad \text{[A7-7]}$$

and the fact that an identity has been obtained without imposing any extra assumption establishes the desired degree of consistency.

The need for us to do the above check can be seen by considering the discussion prior to equations [7-9] and [7-10]. If one chooses not to include the factor of $\sqrt{u_\mu u^\mu}$ in the interaction term of the Lagrangian density [7-8], it is easily shown that the following equation of motion is obtained for the particle instead:

$$\frac{d(mu_\mu)}{d\tau} = \frac{\partial Q}{\partial x_0^\mu} \qquad \text{[A7-8]}$$

Repeating the above consistency check by introducing u^μ on both sides of this new equation, it is then found that the strong condition:

$$\frac{dQ}{d\tau} = 0$$

is deduced instead of the simple identity [A7-7]. Hence choosing the alternative equation of motion [A7-8] and its corresponding Lagrangian density would lead to an unacceptable restriction on the form of the potential Q.

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[3] See, e.g., p. 329 in Goldstein H., *Classical Mechanics*, 2nd Ed. Addison-Wesley, Massachusetts (1980).