Appendix 5: Conservation Difficulty with the Schrodinger Energy-Momentum Tensor

(An appendix to Chapter 6, Section 6.4)

It will be shown here that the energy-momentum tensor in equations [6-24] is not consistent with the conservation condition [6-26].

Inserting expressions [6-24a] and [6-24b]:

$$T^{ij}_{\text{field}} = \frac{\hbar^2}{2m} \{ (\partial^i\psi)(\partial^j\psi^*) + (\partial^i\psi^*)(\partial^j\psi) - g^{ij}(\partial_k\psi^*)(\partial^k\psi) \} - g^{ij}\frac{i\hbar}{2}(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*)$$

$$T^{i0}_{\text{field}} = \frac{i\hbar}{2} \{ \psi^*\partial^i\psi - \psi\,\partial^i\psi^* \}$$

into the left hand side of [6-26]:

$$\partial_j T^{ij}_{\text{field}} + \partial_t T^{i0}_{\text{field}} = -\rho\frac{dp^i}{dt}$$

we obtain:

$$\begin{aligned}
\partial_j T^{ij}_{\text{field}} + \partial_t T^{i0}_{\text{field}} &= \partial_j \Big[\frac{\hbar^2}{2m} \{ (\partial^i\psi)(\partial^j\psi^*) + (\partial^i\psi^*)(\partial^j\psi) - g^{ij}(\partial_k\psi^*)(\partial^k\psi) \} \\
&\qquad - g^{ij}\frac{i\hbar}{2}(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*) \Big] + \partial_t \Big[\frac{i\hbar}{2} \{ \psi^*\partial^i\psi - \psi\,\partial^i\psi^* \} \Big] \\
&= \frac{\hbar^2}{2m} \{ (\partial_j\partial^i\psi)(\partial^j\psi^*) + (\partial^i\psi)(\partial_j\partial^j\psi^*) \\
&\qquad + (\partial_j\partial^i\psi^*)(\partial^j\psi) + (\partial^i\psi^*)(\partial_j\partial^j\psi) \\
&\qquad - (\partial^i\partial_k\psi^*)(\partial^k\psi) - (\partial_k\psi^*)(\partial^i\partial^k\psi) \} \\
&\quad + \frac{i\hbar}{2} \{ -(\partial^i\psi^*)(\partial_t\psi) - \psi^*\partial^i\partial_t\psi \\
&\qquad + (\partial^i\psi)\,\partial_t\psi^*) + \psi\,\partial^i\partial_t\psi^* \\
&\qquad + (\partial_t\psi^*)(\partial^i\psi) + \psi^*\partial_t\partial^i\psi \\
&\qquad - (\partial_t\psi)(\partial^i\psi^*) + \psi\,\partial_t\partial^i\psi^* \}
\end{aligned}$$

which cancels to:

$$\begin{aligned}
\partial_j T^{ij}_{\text{field}} + \partial_t T^{i0}_{\text{field}} &= \frac{\hbar^2}{2m} \{ (\partial^i\psi)(\partial_j\partial^j\psi^*) + (\partial^i\psi^*)(\partial_j\partial^j\psi) \} \\
&\quad + i\hbar \{ -(\partial^i\psi^*)(\partial_t\psi) + (\partial^i\psi)\,\partial_t\psi^*) \} \\
&= (\partial^i\psi^*) \{ \frac{\hbar^2}{2m}\partial_j\partial^j\psi - i\hbar\partial_t\psi \} + (\partial^i\psi) \{ \frac{\hbar^2}{2m}\partial_j\partial^j\psi^* + i\hbar\partial_t\psi^* \} \\
&= (\partial^i\psi^*) \{ -\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi \} + (\partial^i\psi) \{ -\frac{\hbar^2}{2m}\nabla^2\psi^* + i\hbar\partial_t\psi^* \}
\end{aligned}$$

[A5-1]

Hence, using the modified Schrodinger equation [5-23]:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi = -\frac{i\hbar}{2\psi^*}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \}$$

and its complex conjugate:

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + i\hbar\partial_t\psi^* = \frac{i\hbar}{2\psi}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \}$$

equation [A5-1] becomes:

$$\partial_j\ T^{ij}_{field} + \partial_t\ T^{i0}_{field} = (\partial^i\psi^*)\ \{-\frac{i\hbar}{2\psi^*}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} + (\partial^i\psi)\ \{\frac{i\hbar}{2\psi}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \}\ \}$$

This can be written more compactly as:

$$\partial_j\ T^{ij}_{field} + \partial_t\ T^{i0}_{field} = \frac{i\hbar}{2}\ (\frac{\partial^i\psi}{\psi} - \frac{\partial^i\psi^*}{\psi^*})\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \}$$

and using the identity [5-14]:

$$\partial^j S = \frac{\hbar}{2i}\ [\frac{\partial^j\psi}{\psi} - \frac{\partial^j\psi^*}{\psi^*}]$$

we finally have:

$$\partial_j\ T^{ij}_{field} + \partial_t\ T^{i0}_{field} = -\ (\partial^i S)\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} \qquad [A5\text{-}2]$$

Now, identifying $\frac{\nabla S}{m}$ as the velocity of the Bohmian particle, the curly bracket is recognized as corresponding to the continuity equation describing the conservation of the matter making up this particle. This allows us to demonstrate that [A5-2] is not consistent with [6-26] by considering the usual, non-relativistic case of no particle creation or annihilation. In this case the curly bracket is zero and so equation [A5-2] reduces simply to

$$\partial_j\ T^{ij}_{field} + \partial_t\ T^{i0}_{field} = 0$$

This result prohibits exchanges of energy and momentum between the field and particle and so is not compatible with equation [6-26].