Appendix 3: Rate of Change of a Particle's Energy in a Scalar Field

(An appendix to Chapter 6, Section 6.3.1)

Equation [6-12] will be derived here. Starting from the fact that total energy equals kinetic energy plus potential energy, we can write:

$$\frac{dE_{\text{particle}}}{dt} = \frac{d}{dt}(KE_{\text{particle}} + PE_{\text{particle}})$$

$$= \frac{d}{dt}\left(\frac{p^2}{2m} + q\phi\right)$$

$$= \frac{1}{2m}\frac{dp^i}{dt}\frac{d}{dp^i}(-p_j p^j) + q\left(\frac{dx_0^i}{dt}\frac{\partial\,\phi(\mathbf{x_0})}{\partial x_0^i} + \frac{\partial\,\phi(\mathbf{x_0})}{\partial t}\right)$$

$$= -\frac{1}{2m}\frac{dp^i}{dt}(g_{ij}p^j + p_j\delta_i^j) + q\left(\frac{p^i}{m}\partial_i\phi + \frac{\partial\phi}{\partial t}\right)$$

$$= -\frac{1}{m}\frac{dp^i}{dt}p_i + q\left(\frac{p_i}{m}\partial^i\phi + \frac{\partial\phi}{\partial t}\right)$$

Therefore, using equation [6-11], we have:

$$\frac{dE_{\text{particle}}}{dt} = -\frac{1}{m}(q\partial^i\phi)p_i + q\left(\frac{p_i}{m}\partial^i\phi + \frac{\partial\phi}{\partial t}\right)$$

and cancelling yields:

$$\frac{dE_{\text{particle}}}{dt} = q\frac{\partial\phi}{\partial t}$$

which is equation [6-12].