Appendix 2: Velocity Expression corresponding to the Modified Schrodinger Equation

(An appendix to Chapter 5, Section 5.3)

It will be shown here that the velocity expression $\mathbf{v} = \nabla S/m$ corresponding to the current density of the standard Schrodinger equation remains unchanged in going to the modified Schrodinger equation [5-23]:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi = -\frac{i\hbar}{2\psi^*}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} \qquad \text{[A2-1]}$$

We will follow the usual steps involved in deriving the Schrodinger continuity equation. The conjugate equation to [A2-1] is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + i\hbar\partial_t\psi^* = \frac{i\hbar}{2\psi}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} \qquad \text{[A2-2]}$$

Multiplying [A2-1] by ψ^* and [A2-2] by ψ then subtracting the two results, we have

$$\psi^*(-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\partial_t\psi) - \psi\,(-\frac{\hbar^2}{2m}\nabla^2\psi^* + i\hbar\partial_t\psi^*)$$
$$= -\frac{i\hbar}{2}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \} - \frac{i\hbar}{2}\ \{\nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho\ \}$$

which simplifies to:

$$\psi^*\frac{\hbar}{2im}\nabla^2\psi - \psi\,\frac{\hbar}{2im}.\nabla^2\psi^* + \psi^*\partial_t\psi + \psi\partial_t\psi^* = \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho$$

i.e.,

$$\frac{\hbar}{2im}\nabla.(\psi^*\nabla\psi - \psi\nabla\psi^*) + \partial_t(\psi^*\psi) = \nabla.(\frac{\rho\nabla S}{m}) + \partial_t\rho \qquad \text{[A2-3]}$$

Expressing the wave function ψ in the form:

$$\psi = R\ \exp\,(\frac{iS}{\hbar})$$

equation [A2-3] can be written as:

$$\nabla.(R^2\frac{\nabla S}{m}) + \partial_t(R^2) = \nabla.(\rho\frac{\nabla S}{m}) + \partial_t\rho$$

which rearranges to:

$$\nabla\cdot[(R^2-\rho)\frac{\nabla S}{m}]+\partial_t(R^2-\rho)=0$$

This can be recognized as a continuity equation containing a current density $(R^2-\rho)$ and a flow velocity $\frac{\nabla S}{m}$. Now, although the current density expression is different from the usual Schrodinger one, the velocity expression is the same. It has therefore been shown that our modified Schrodinger equation yields the same velocity expression as does the standard Schrodinger equation.